Exercises

5.16 **Fractal dimensions.**[1 2] (Mathematics, Complexity) ④

There are many strange sets that emerge in science. In statistical mechanics, such sets often arise at continuous phase transitions, where self-similar spatial structures arise (Chapter 12). In chaotic dynamical systems, the attractor (the set of points occupied at long times after the transients have disappeared) is often a fractal (called a *strange attractor*). These sets are often tenuous and jagged, with holes on all length scales; see Figs 12.2, 12.5, and 12.11.

We often try to characterize these strange sets by a dimension. The dimensions of two extremely different sets can be the same; the path exhibited by a random walk (embedded in three or more dimensions) is arguably a two-dimensional set (note 6 on p. 17), but does not locally look like a surface. However, if two sets have different spatial dimensions (measured in the same way) they are certainly qualitatively different.

There is more than one way to define a dimension. Roughly speaking, strange sets are often spatially inhomogeneous, and what dimension you measure depends upon how you weight different regions of the set. In this exercise, we will calculate the *information dimension* (closely connected to the non-equilibrium entropy), and the *capacity dimension* (originally called the *Hausdorff dimension*, also sometimes called the *fractal dimension*).

To generate our strange set—along with some more ordinary sets—we will use the logistic map[3]

$$f(x) = 4\mu x(1-x). \tag{1}$$

The attractor for the logistic map is a periodic orbit (dimension zero) at $\mu = 0.8$, and a chaotic, cusped density filling two intervals (dimension one)[4] at $\mu = 0.9$. At the onset of chaos at $\mu = \mu_\infty \approx 0.892486418$ (Exercise 12.9) the dimension becomes intermediate between zero and one; this strange, self-similar set is called the *Feigenbaum attractor*.

Both the information dimension and the capacity dimension are defined in terms of the occupation P_n of cells of size ϵ in the limit as $\epsilon \to 0$.

(a) *Write a routine which, given μ and a set of bin sizes ϵ, does the following.*

- *Iterates f hundreds or thousands of times (to get onto the attractor).*
- *Iterates f a large number $N_{\rm tot}$ more times, collecting points on the attractor. (For $\mu \le \mu_\infty$, you could just integrate 2^n times for n fairly large.)*
- *For each ϵ, use a histogram to calculate the probability P_j that the points fall in the jth bin.*
- *Return the set of vectors $P_j[\epsilon]$.*

You may wish to test your routine by using it for $\mu = 1$ (where the distribution should look like $\rho(x) = 1/\pi\sqrt{x(1-x)}$, Exercise 4.3(b)) and $\mu = 0.8$ (where the distribution should look like two δ-functions, each with half of the points).

The capacity dimension. The definition of the capacity dimension is motivated by the idea that it takes at least

$$N_{\rm cover} = V/\epsilon^D \tag{2}$$

bins of size ϵ^D to cover a D-dimensional set of volume V.[5] By taking logs of both sides we find

[1] From *Statistical Mechanics: Entropy, Order Parameters, and Complexity* by James P. Sethna, copyright Oxford University Press, 2007, page 101. A pdf of the text is available at pages.physics.cornell.edu/sethna/StatMech/ (select the picture of the text). Hyperlinks from this exercise into the text will work if the latter PDF is downloaded into the same directory/folder as this PDF.

[2] This exercise and the associated software were developed in collaboration with Christopher Myers.

[3] We also study this map in Exercises 4.3, 5.9, and 12.9.

[4] See Exercise 4.3. The chaotic region for the logistic map does not have a strange attractor because the map is confined to one dimension; period-doubling cascades for dynamical systems in higher spatial dimensions have fractal, strange attractors in the chaotic region.

[5] Imagine covering the surface of a sphere in 3D with tiny cubes; the number of cubes will go as the surface area (2D volume) divided by ϵ^2.

$\log N_{\mathrm{cover}} \approx \log V + D \log \epsilon$. The capacity dimension is defined as the limit

$$D_{\mathrm{capacity}} = \lim_{\epsilon \to 0} \frac{\log N_{\mathrm{cover}}}{\log \epsilon}, \tag{5.44}$$

but the convergence is slow (the error goes roughly as $\log V / \log \epsilon$). Faster convergence is given by calculating the slope of $\log N$ versus $\log \epsilon$:

$$\begin{aligned} D_{\mathrm{capacity}} &= \lim_{\epsilon \to 0} \frac{\mathrm{d} \log N_{\mathrm{cover}}}{\mathrm{d} \log \epsilon} & (5.45) \\ &= \lim_{\epsilon \to 0} \frac{\log N_{j+1} - \log N_j}{\log \epsilon_{i+1} - \log \epsilon_i}. & (3) \end{aligned}$$

(b) *Use your routine from part (a), write a routine to calculate $N[\epsilon]$ by counting non-empty bins. Plot D_{capacity} from the fast convergence eqn 5.45 versus the midpoint $\frac{1}{2}(\log \epsilon_{i+1} + \log \epsilon_i)$. Does it appear to extrapolate to $D = 1$ for $\mu = 0.9$?*[6] *Does it appear to extrapolate to $D = 0$ for $\mu = 0.8$? Plot these two curves together with the curve for μ_∞. Does the last one appear to converge to $D_1 \approx 0.538$, the capacity dimension for the Feigenbaum attractor gleaned from the literature? How small a deviation from μ_∞ does it take to see the numerical cross-over to integer dimensions?*

Entropy and the information dimension. The probability density $\rho(x_j) \approx P_j/\epsilon = (1/\epsilon)(N_j/N_{\mathrm{tot}})$. Converting the entropy formula 5.20 to a sum gives

$$\begin{aligned} S &= -k_B \int \rho(x) \log(\rho(x))\,\mathrm{d}x \\ &\approx -\sum_j \frac{P_j}{\epsilon} \log\left(\frac{P_j}{\epsilon}\right) \epsilon \\ &= -\sum_j P_j \log P_j + \log \epsilon & (4) \end{aligned}$$

(setting the conversion factor $k_B = 1$ for convenience).

You might imagine that the entropy for a fixed-point would be zero, and the entropy for a period-m cycle would be $k_B \log m$. But this is incorrect; when there is a fixed-point or a periodic limit cycle, the attractor is on a set of dimension zero (a bunch of points) rather than dimension one. The entropy must go to minus infinity—since we have precise information about where the trajectory sits at long times. To estimate the 'zero-dimensional' entropy $k_B \log m$ on the computer, we would use the discrete form of the entropy (eqn 5.19), summing over bins P_j instead of integrating over x:

$$S_{d=0} = -\sum_j P_j \log(P_j) = S_{d=1} - \log(\epsilon). \tag{5}$$

More generally, the 'natural' measure of the entropy for a set with D dimensions might be defined as

$$S_D = -\sum_j P_j \log(P_j) + D \log(\epsilon). \tag{5.48}$$

Instead of using this formula to define the entropy, mathematicians use it to define the information dimension

$$D_{\mathrm{inf}} = \lim_{\epsilon \to 0} \left(\sum P_j \log P_j\right) / \log(\epsilon). \tag{6}$$

The information dimension agrees with the ordinary dimension for sets that locally look like $\mathbb{R}^D$. It is different from the capacity dimension (eqn 5.44), which counts each occupied bin equally; the information dimension counts heavily occupied parts (bins) in the attractor more heavily. Again, we can speed up the convergence by noting that eqn 5.48 says that $\sum_j P_j \log P_j$ is a linear function of $\log \epsilon$ with slope D and intercept S_D. Measuring the slope directly, we find

$$D_{\mathrm{inf}} = \lim_{\epsilon \to 0} \frac{\mathrm{d} \sum_j P_j(\epsilon) \log P_j(\epsilon)}{\mathrm{d} \log \epsilon}. \tag{5.50}$$

(c) *As in part (b), write a routine that plots D_{inf} from eqn 5.50 as a function of the midpoint $\log \epsilon$, as we increase the number of bins. Plot the curves for $\mu = 0.9$, $\mu = 0.8$, and μ_∞. Does the information dimension agree with the ordinary one for the first two? Does the last one appear to converge to $D_1 \approx 0.517098$, the information dimension for the Feigenbaum attractor from the literature?*

Most 'real world' fractals have a whole spectrum of different characteristic spatial dimensions; they are *multifractal*.

[6] In the chaotic regions, keep the number of bins small compared to the number of iterates in your sample, or you will start finding empty bins between points and eventually get a dimension of zero.