RG flows from non-perturbative coarse graining

(Sethna, “Entropy, Order Parameters, and Complexity”, ex. XXX)

This exercise is primarily analytical: only those parts with computational components are included in this file. The exercise will be available at https://sethna.lassp.cornell.edu/StatMech/SethnaExercises.pdf.

Long intro to non-perturbative RG and momentum-space RG...

Sparing ourselves this calculation, we shall quote the LPA approximation for the coarse-graining equations up to quartic terms:
$dF/dl = -k^3/(1+(A/k^2))$
$dA/dl = 3Gk/(1+(A/k^2))^2$
$dG/dl = -18(G^2/k)/(1+(A/k^2)^3$
$dk/dl = -k$

These are not yet the renormalization-group flow equations: we have yet to rescale properly to see the fixed point. But we can use these as if we were the experimentalist, measuring universal power laws in the “laboratory” provided by $V\infty(M\infty,A\infty,G\infty,F\infty)$.

We shall focus on the transition for $G0=0.2$, where the critical point lies at $Ac0\approx-0.19053$. At the critical point, the macroscopic potential develops a double well, leading to a finite magnetization. Hence we expect $A\infty=0$ when $A0$ equals $Ac0$.

```
V[M_, F_, A_, G_] := F + ... + (G / 2) M^4

CoarseGrainingFlowEqns =
 {k'[ell] == -k[ell],
  A'[ell] == 3 G[ell] × k[ell] / (1 + (A[ell] / k[ell]^2))^2,
  G'[ell] == -18 (G[ell]^2 / k[ell]) / (1 + A[ell] / k[ell]^2)^3,
  F'[ell] == ...}
```

(a) Numerically solve the flow equations starting at $l=0$, $k=1$, $A0=Ac0$, $G0=0.2$, and $F0=0$, evaluating them at $l\in[0,0.25,0.5,1,2,4]$. Plot the potentials $V\_l\,(M)$. Does the potential appear to be at the critical point, where the bulk quadratic term $A\infty=0$? Note also that the microscopic energy $V0$ has a double well, but the well disappears under coarse-graining. Why should this be expected, for a low enough barrier? (Hint: The barrier height in $V0$ is measured in units of the temperature. If it is much less than one, do you expect the system to stay in one well?)

```
Ac = -0.19053;
CoarseGrainingInitialConditions = {A[0] == Ac, G[0] == ..., F[0] == 0, k[0] == 1};
eqns = Join[CoarseGrainingFlowEqns, CoarseGrainingInitialConditions];
ellFinal = 4;
ells2Plot = {0, 0.25, 0.5, ...};
solCritical =
  NDSolve[eqns, {A[ell], ..., F[ell], k[ell]}, {ell, 0, ellFinal}][[1]];
CurvesCritical = Table[V[M, F[ell], A[ell], ...] /. solCritical, {ell, ells2Plot}];
Plot[CurvesCritical, {M, -4, 4}, PlotRange → {-0.5, 0.5},
 PlotLegends → LineLegend[Table[ell, {ell, ells2Plot}]]]
```

Your answer here (or in a separate writeup).

Since our free energy $V(M)$ is measured in units of the temperature, raising the temperature lowers A, G, and F. As A directly controls the development of magnetization, let us mimic the experimental temperature by varying $A0$ through the critical value $Ac0$. We shall confine ourselves to the approach to $Ac0$ from above; more sophisticated NPFRG methods are needed to behave properly below the transition temperature.

Let us measure the macroscopic susceptibility χ=∂Mbar /∂H | H=0, where Mbar minimizes $V\infty(M)$–MH.

(b) Show that χ=1/(2$A\infty$) in the paramagnetic single-well phase when $A0$>$Ac0$. (Hint: Find the equation satisfied by Mbar, and take its derivative with respect to H.)

Your answer here (or in a separate writeup).

(c) As in part (a), plot the potential V_*l* as it evolves starting at $Ac0$+0.05. Does it converge to a fixed potential, representing the bulk free energy? Measure the bulk quadratic term $A\infty$ in the free energy for a range of microscopic values $A0$ close to $Ac0$, and estimate γ. (Go to larger *l* as your initial temperature gets close to $Ac0$. You can do a power-law fit, but also you can just do a log-log plot of $\chi(A0)\times(A0$-$Ac0)$^γtry vs. $A0$–$Ac0$ and vary γtry until it has a flat region.) Compare your value γtry to the value derived from conformal bootstrap, γboot~1.237075.

```
CoarseGrainingInitialConditions = {A[0] == Ac + 0.05, G[0] == ...};
eqns = Join[CoarseGrainingFlowEqns, CoarseGrainingInitialConditions];
solCritical = NDSolve[ ...][[1]];
CurvesCritical = Table[ ..., {ell, ells2Plot}];
Plot[ ...]
```

```
In[ ]:= Clear[CoarseGrainingInitialConditions, eqnsVaryA, sols]
```

```
deltaAs = PowerRange[10^-4, 10, 10^(0.1)];
AInits = Ac + deltaAs;

eqnsVaryA[n_] := Join[CoarseGrainingFlowEqns,
  {A[0] == AInits⟦n⟧, G[0] == 0.2, F[0] == ..., k[0] == ...}]
ellFinal = 4;
sols = Table[
   NDSolve[eqnsVaryA[n], {A[ell], G[ell], F[ell], k[ell]}, {ell, 0, ellFinal}]⟦1⟧,
   {n, 1, Length[AInits]}];
chis = .../(...A[ell]) /. sols /. ell → ellFinal;
γBootstrap = 1.237075;
(* Vary γTry until plot is mostly horizontal *)
γTry = 0
points = Transpose[{deltaAs, chis deltaAs^γTry}];
ListLogLogPlot[points]
```

Your answer here (or in a separate writeup).

Discussion ...

These rescaled equations for $d=3$ are (finally!) our RG flow equations
$da/dl=2a+3g/(1+a)^2$
$dg/dl=g-18g^2/(1+a)^3$
$df/dl=3f-1/(1+a)$.

Part (d) deriving these, part (e) finding the fixed point...

What are $ac0$ and $g0$, the rescaled parameters corresponding to the experiments we performed in parts(a) and(c)? Since l starts at zero with the microscopic Hamiltonian, $k\_l=1$, and thus the coarse-graining transition you studied happen at RG-flow initial conditions $g0=G0=0.2$ and $ac0=Ac0\approx-0.1905316$.

(f) Launch trajectories near this point, varying $a0$ slightly above and below $ac0$, and plot the trajectories in the $a0,g0$ plane. Show that they pass near to the fixed point before veering off to high or low rescaled temperatures.

```
RGFlowEqns =
  {a'[ell] == 2 a[ell] + 3 g[ell] / (1 + a[ell])^2,
   g'[ell] == g[ell] - 18 g[ell]^2 / (1 + a[ell])^3,
   f'[ell] == ...};
```

```
aStar = -1/13.; gStar = 96/2197.; fStar = ...;
```

```
deltaas = Range[-0.025, 0.025, 0.005];
aInits = Ac + deltaas;
ellFinal = 2;
eqnsVarya[n_] := Join[RGFlowEqns, {a[0] ⩵ aInits⟦n⟧, g[0] ⩵ ..., f[0] ⩵ ...}]
sols = Table[NDSolve[eqnsVarya[n], {a[ell], g[ell], f[ell]}, {ell, 0, ellFinal},
      AccuracyGoal → 13, PrecisionGoal → 13]⟦1⟧, {n, 1, Length[aInits]}];
flows = {g[ell], a[ell]} /. sols;
flowPlot =
  ParametricPlot[flows, {ell, 0, ellFinal}, PlotRange → {{0, 0.25}, {-0.2, 0.05}}];
pointsPlot =
  ListPlot[{{gStar, aStar}, {0.2, Ac}}, PlotStyle → {Red, PointSize[0.02]}];
Show[pointsPlot, flowPlot]
```

The next step is to linearize the flows for a and g at the fixed point $(a*,g*)$, to find the Jacobian $J \equiv ((\partial \dot{a}/\partial a\ \ ,\ \partial \dot{g}/\partial a),\ (\partial \dot{a}/\partial g\ \ ,\ \partial \dot{g}/\partial g))$ = ((5/3, 24/169), (169/48, −1))
where, e.g., $\dot{a} = \partial a/\partial l$.

(g) Calculating all the components of J is straightforward, but a bit tedious. Verify that the lower right element $\partial \dot{g}/\partial g$ = −1. Then use J to numerically find the eigenvalues and right eigenvectors. Add these to your plot of part (e), and verify that the flows approach the fixed point along the irrelevant eigendirection, and then veer away along the relevant eigendirection.

Your answer here (or in a separate writeup).

Warning: Eigenvectors are in (a,g,f) order. We want to plot (g, a).

```
J = {{5. / 3., 169. / 48.}, {24. / 169., ...}};
{vals, vecs} = Eigensystem[J]
{λT, λU} = vals
{vecT, vecU} = vecs

eigPlot = ListPlot[{{{gStar, aStar}, {gStar + vecT⟦2⟧, aStar + vecT⟦1⟧}},
     {{gStar, aStar}, {gStar + vecU⟦2⟧, aStar + vecU⟦1⟧}}}, Joined → True,
   PlotRange → {{0, 0.25}, {-0.2, 0.1}}, PlotStyle → Black];
Show[pointsPlot, flowPlot, eigPlot]
```

Calculating predicted critical exponents: parts (h), (i), (j), (k)