Proposition:

The following 2-dimensional example of Origami microstructure cannot be folded along all the marked edges to one plane:

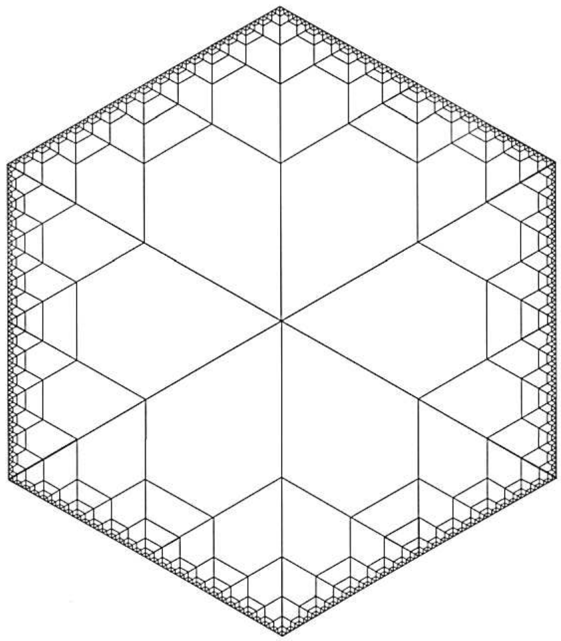

Fig. 1 2-D Origami microstructure

To prove this, for simplicity we prove the following pattern with 1/6 of the original area with level-3 self-similarity is not foldable (which is a stronger claim than the original):

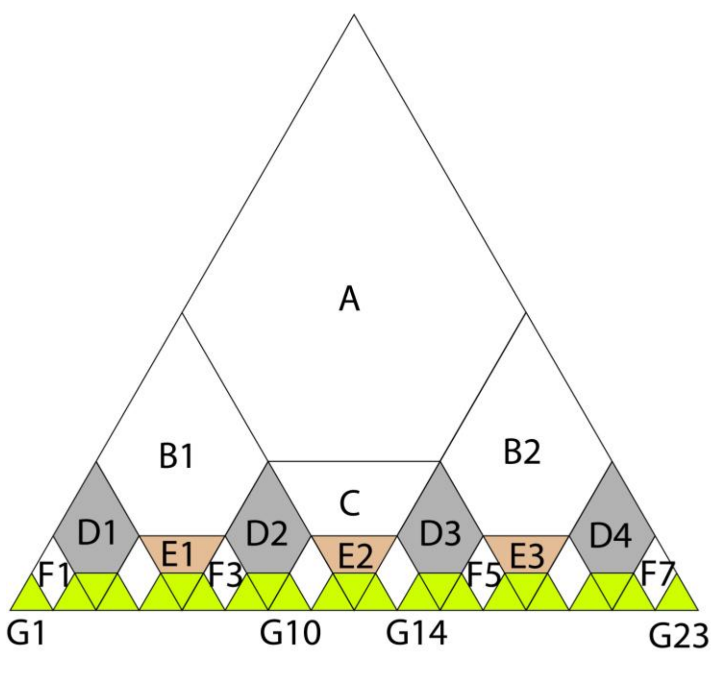

Fig.2 Labeling of all the domains

Proof:

All the nodes of this pattern are 4-edge nodes, and the opposite opening angles add up to 180 degree, therefore there exists a mapping that projects this pattern to a "folded state" on the plane (by the theorem by Prof. James). If we label all the "domains" by A, B1, B2, …. , G1, G2, …. , G23 (as in Fig.2), and fix the position, rotation and facing (remains facing up) of the domain A during the projection, the positions and orientations of all the other domains can be uniquely determined. For example, the domain B1 in the folded state will be facing down, rotated and translated to occupy the space shown as the yellow area in the right panel of Fig.2. It is important to note that all the small triangles (G1, G2, …. , G23) after the projection will overlap with the same triangular area ("focus triangle") enclosed by the red line in the right panel of Fig.2 with varied orientations and facings, and all the other domains (A, B1, B2, …., F1, F2, …. , F8) cover this focal triangle as well.

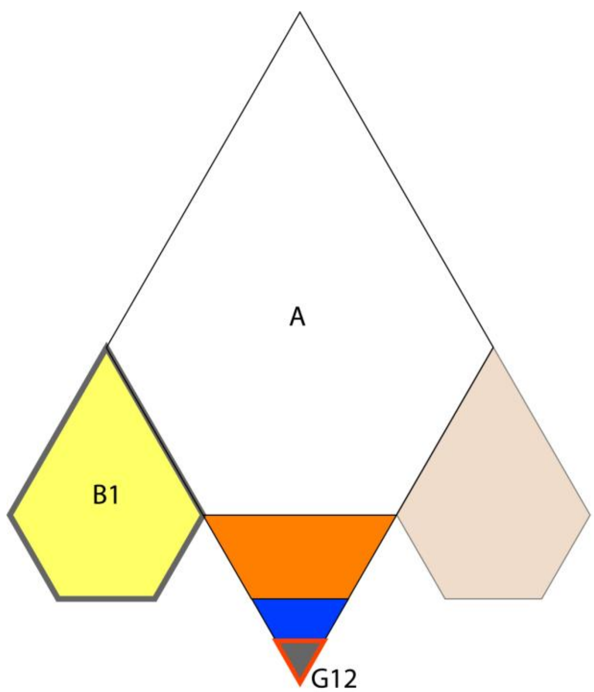

Fig.3 The projected folded state (right panel) of a few selected domains before folding (left panel).

Despite the viability of this projection to a 2D plane, for a real folded piece of paper one has to be able to identify the sequence of the layer stack wherever overlap occurs, i.e. which domain is on top of which. For this particular pattern we note that all the domains cover the "focal triangle" in the folded state, therefore a folded state of the real piece of paper (if possible at all) could be identified by a particular order (permutation) of all the domains. This permutation has to satisfy the following constraint: if domain X, Y and Z appear in the permutation with the order …, X, …, Y, …, Z, …, and X and Z share an edge (X-Z), the projected position of the edge (X-Z) in the folded state has to be outside (or at the boundary of) the projected area of domain Y. Otherwise the piece of paper (in domain Y) would have to penetrate itself (at the edge X-Z). With this rule in mind, let's try to construct a permutation that represents a possible folded state.

Since the G domains are all projected into the focal triangle, all the edges related to the G domains are projected to one of the three edges of the focal triangle which we call focal edges thereafter. These focal edges include the edges among G domains, edges between G domains and F domains, edges between G domains and E domains and edges between G domains and D domains. On the other hand, the focal triangle is fully within the boundary of domains A, B1, B2, and C, which forbids A, B1, B2 or C appearing between any two domains sharing a focal edge in the permutation. (For example looking at Fig.3, since E2 and G12 share a focal edge, they have to placed on the same side of B1 after folded. Therefore a sequence of ..., E2, …, B1, …, G12, … is illegal.) Because all the D, E, F, G domains are interconnected by focal edges, they form a cluster in any viable permutation, and the rest domains (A, B1, B2 and C) cannot be inserted into the cluster. Therefore, the folded state of the paper has to be represented by a permutation of A, B1, B2, C and Q with Q the cluster containing all the D, E, F, G domains.

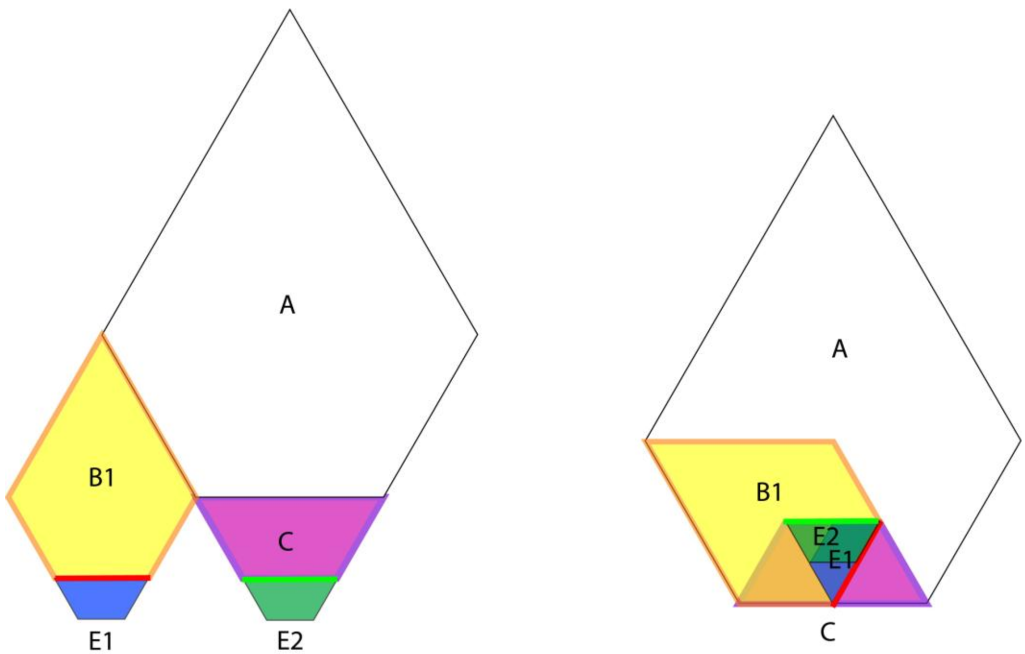

Fig. 4 Illustration of the folded state where B1-E1 edge is projected inside the area of C, and C-E2 edge is inside the area of B1.

Note that the B1-E1 edge (highlighted in red in Fig.4) lies inside the area of C in the folded state and E1 is a member of Q, so C cannot be placed between B1 and Q in the permutation. Note that the C-E2 edge (highlighted in green in Fig. 4) lies inside the area of B1, so B1 cannot be placed between C and Q. Therefore B1, C and Q have to appear in the permutation in the order of B1, Q, C (or its reverse). Similarly the same rule applies to B2, Q and C, and therefore the viable permutations (without considering A) are limited to B1, B2, Q, C or B2, B1, Q, C (or their reverse). However, in the folded state the B1-D1 edge lies inside the area of B2 (not shown), making the B1, B2, Q, C permutation illegal; while the B2-D4 edge lies inside the area of B1 (not shown), making the B2, B1, Q, C permutation illegal. Therefore no permutation of all the domains can satisfy the imposed constraint.

In summary, there is no viable way to stack the domains and avoid penetration of the paper through a crease, so it is not possible to fold the paper to the hypothetically projected 2D “folded” state.