Physics 218—Exam I (October 5) **Fall 2002**
James P. Sethna

NAME:______________________________

The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

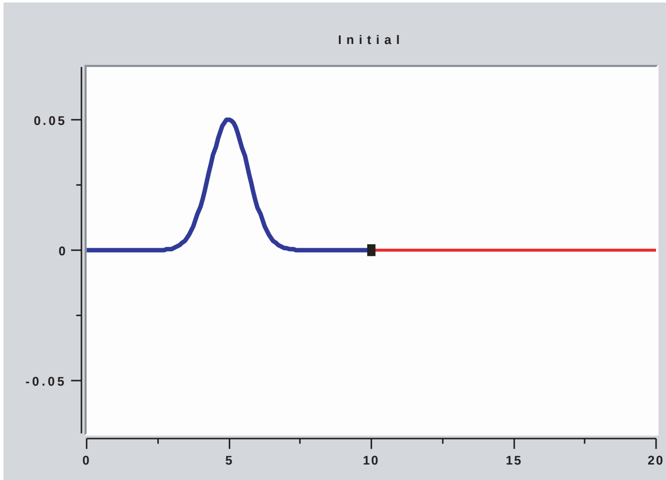
Multiple Choice (48 pts):____ **x 12pts =** ________________

Short Answer: **S1 (52 pts)** ________________

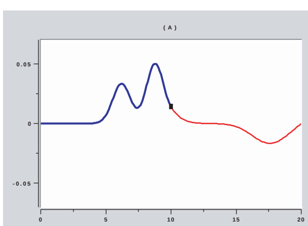
TOTAL ________________

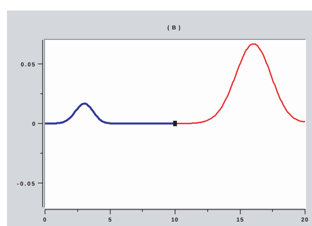
Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

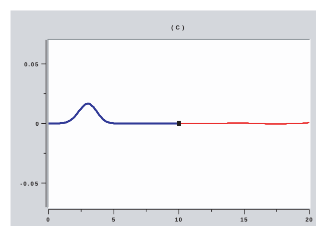
M1. (12 pts) Reflection, Transmission, and Boundary Conditions.

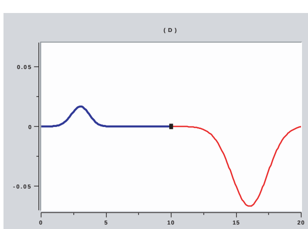


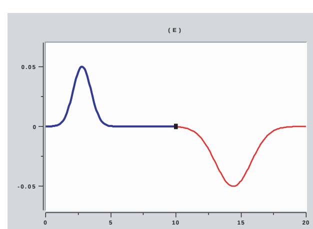
The string shown above has speed of sound $c_1 = 5$m/s on the left and $c_2 = 10$m/s on the right. The boundary conditions on the right end are **fixed** at $L = 20$m; the left end is at $x = 0$m, and the massless knot connecting the two strings is halfway, at $x = 10$m. A pulse at $t = 0$ centered at $x_0 = 5$m is traveling to the right. The transmitted pulse is observed to hit the end of the string ($x = L$) at about 2 seconds. Which figure below correctly represents the pulse at time $t = 2.4$ seconds? (All graphs are on the same scale.)

(A) 

(B) 

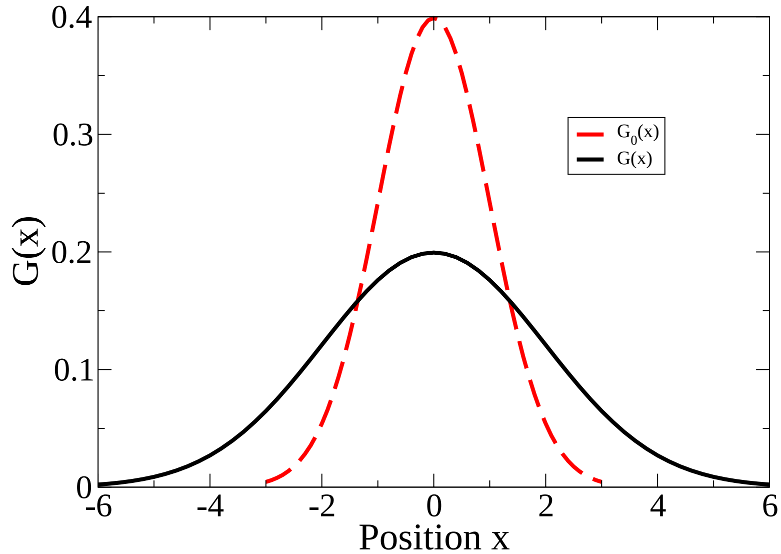
(C) 

(D) 

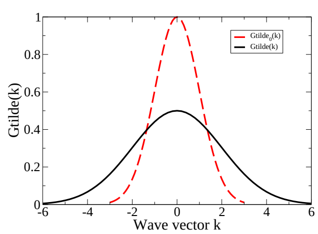
(E) 

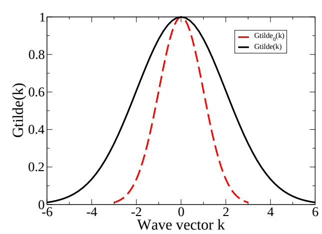
Answer

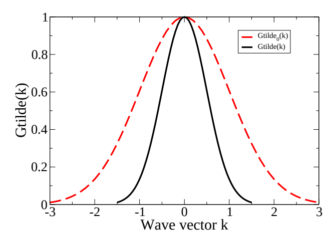
M2. (12 pts) Fourier Series and Uncertainty.

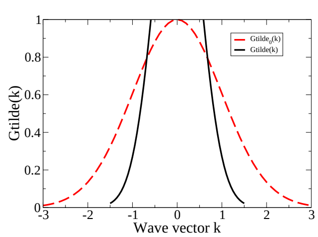


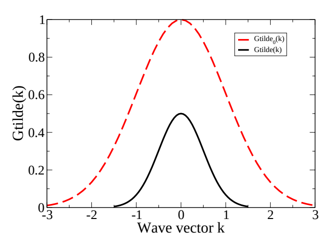
The dashed line above shows $G_0(x) = 1/\sqrt{2\pi}\,\exp(-x^2/2)$. The dark line shows another function $G(x)$. The areas under the two curves $G(x)$ and $G_0(x)$ are the same. The dashed lines in the choices below represent the Fourier transform $\tilde{G}_0(k) = \exp(-k^2/2)$. Which solid curve below represents the Fourier transform of the black curve above?

(A) 

(B) 

(C) 

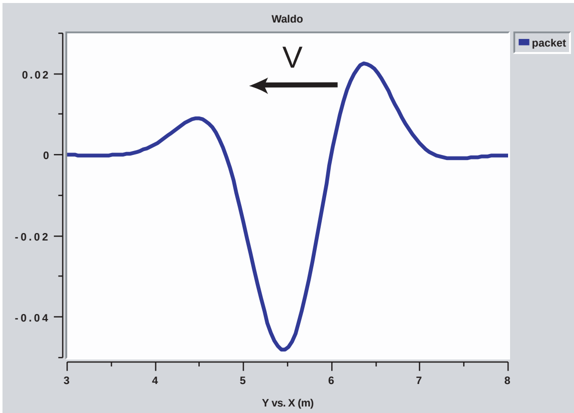
(D) 

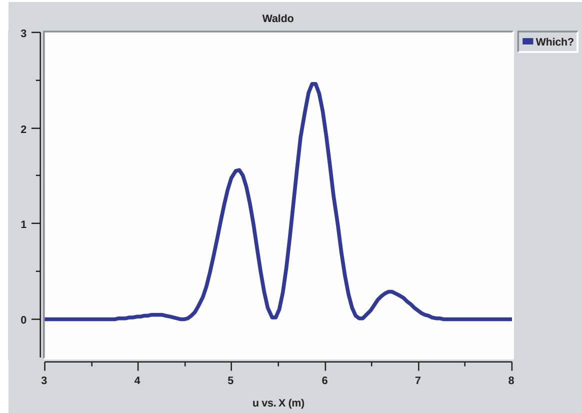
(E) 

Answer

Beware: the Fourier lab program auto-rescaled the height.

M3. (12 pts) Energy, Power, Momentum, Velocity.





On the left is the current shape $\eta(x,0)$ of a packet at time $t=0$, traveling to the *left* on a string of mass density $\lambda_0 = 0.1\text{kg/m}$ and tension $\tau = 160\text{N}$. What is the quantity shown on the graph at right?

(A) Chunk velocity $\partial\eta/\partial t$.

(B) Total energy density $1/2\lambda_0(\partial\eta/\partial t)^2 + 1/2\tau(\partial\eta/\partial x)^2$.

(C) Power $-\tau\partial\eta/\partial x\,\partial\eta/\partial t$.

(D) Momentum $-\lambda_0\partial\eta/\partial x\,\partial\eta/\partial t$.

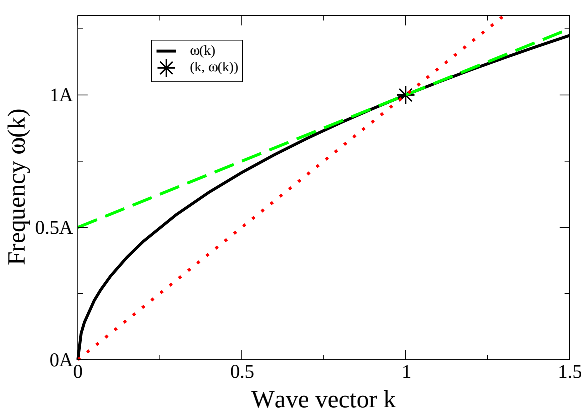
(E) Time independent, $\tilde{\eta}(x,t) = \eta(x, t-\Delta)$.

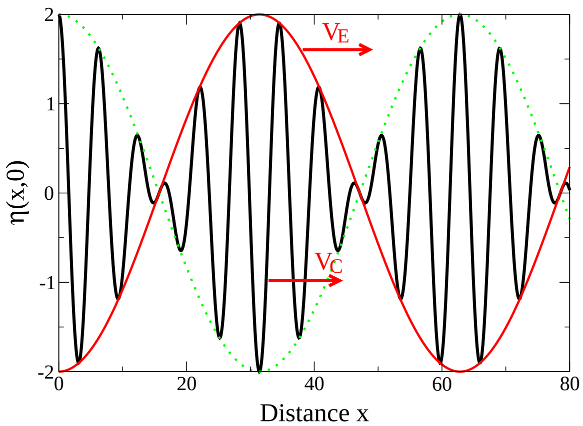
(F) Sideways motion, $\tilde{\eta}(x,t) = \eta(x,t) + \Delta$.

Answer

Hint: Which quantities are negative for a pulse moving left?

M4. (12 pts) Group and Phase Velocities.





An unusual crystal has a dispersion relation $\omega[k] = A\sqrt{k}$ (left figure above). It is vibrating with a superposition of two rightward-moving sinusoidal traveling waves with nearly the same wavevector $k \pm \delta k$:

$$\eta(x,t) = \cos\left((k+\delta k)x - \omega[k+\delta k]t\right) + \cos\left((k-\delta k)x - \omega[k-\delta k]t\right).$$

At $t = 0$ the superposition has the form shown above on the right: it consists of an envelope wave traveling to the right at a velocity v_E and a carrier wave traveling to the right at velocity v_C. Which of the following gives the relation of v_E to v_C for this system? (You may ignore corrections of order $(\delta k)^2$.

(A) $v_E = v_C$ as $k \to 0$

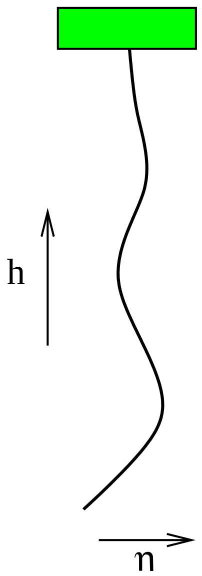
(B) $v_E = 2v_C$

(C) $v_E = v_C/2$

(D) $v_E[k] \sim \sqrt{2 - 2\cos(ka)}$ and $v_C[k] \sim -\sqrt{2 - 2\cos(ka)}$

(E) $\partial^2 v_E/\partial t^2 = c^2 \partial^2 v_C/\partial x^2$.

Answer



Short Answer: Show Your Work

S1. (52 pts) Wave Equation under Gravity.

A string of mass density λ_0 dangles from the ceiling. Height h is measured from the bottom of the string: the horizontal displacement is $\eta(h)$ (above).

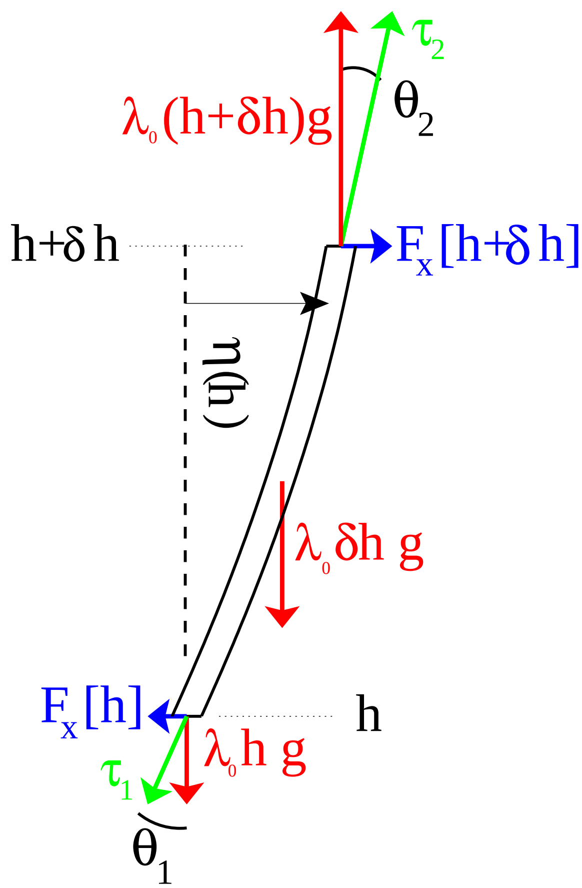
Gravity acting on the string breaks two symmetries of the ordinary wave equation. First, the string no longer has reflection symmetry along its length: under the change $h \to -h$ the top and bottom of the string are interchanged. Second, the string is no longer homogeneous along its length: the upper parts of the string are under higher tension than the lower parts (since the former need to support the weight of the latter).

(A) **Broken Symmetries.** (12 points)
Which of the following constants (A,B,C,D,E,F) that were not allowed in the ordinary wave equation are now allowed to be non-zero by symmetry?

$$\partial^2\eta/\partial t^2 = A + Bt + C\eta + D\partial\eta/\partial h + E\partial\eta/\partial t + (c^2 + Fh)\partial^2\eta/\partial h^2. \qquad (S.1)$$

Constants allowed =

(B) (20 points)



The free body diagram for a chunk of the dangled string is shown above. We assume that the chunk motion is purely transverse (in the x direction) so the vertical forces must add to zero, as shown. Give the formulas for the horizontal components of the forces on the two ends of the chunk, in terms of λ_0, g, h, δh, θ_1, and θ_2.

$F_x[h] =$

$F_x[h + \delta h] =$

Did you leave your answer in terms of θ_1 and θ_2? Wait until part (C) to convert to derivatives.

(C) (20 points)

Write the equation of motion for the dangling string, by setting the total horizontal force from part (B) equal to ma for the chunk. Write your answer in terms of λ_0, g, h, $\partial\eta/\partial h$, and $\partial^2\eta/\partial h^2$; it should not involve δh, θ_1, or θ_2.

$\partial^2\eta/\partial t^2 =$

Hint: your answer might involve terms that are allowed by symmetry in part (A).

Physics 218—Exam I (October 7) Fall 2002

Formula Sheet

James P. Sethna

Complex Trigonometry. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$. $\sin(A+B) = \sin A \cos B + \cos A \sin B$, $\cos(A+B) = \cos A \cos B - \sin A \sin B$, $\exp(iA) + \exp(iB) = \exp(i(A+B)/2)\, 2\cos((A-B)/2)$, $\cos(A) + \cos(B) = 2\cos((A+B)/2)\cos((A-B)/2)$.

Fourier.
$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx$ where $k_m = 2\pi m/L$; $y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x)$.
$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x)\exp(-ikx)\,dx$; $y(x) = (1/2\pi)\int_{-\infty}^{\infty} \tilde{y}(k)\exp(ikx)\,dk$.
$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N)$ with $m = 0, \ldots N-1$; $\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-ik_m x_\ell)$.
$G(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-(x-x_0)^2/2\sigma^2\right)$; $\tilde{G}(k) = \exp(-ikx_0)\exp(-\sigma^2 k^2/2)$.

Orthonormality.
$(1/L)\int_0^L \exp(ik_m x)\exp(ik_n x)\,dx = \delta_{mn}$, where $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$.
The Fourier transform of a Gaussian of width σ, $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-x^2/2\sigma^2\right)$ is a Gaussian $\tilde{f}(k) = \exp\left(-\sigma^2 k^2/2\right)$ of width $1/\sigma$. The Fourier transform of $f(x-x_0)$ is $\exp(-ikx_0)\tilde{f}(k)$.

Wave Equation $\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$. Traveling wave solution $\eta(x,t) = f(x \pm ct)$. Standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$. Traveling sine wave (Plane wave) $\eta(x,t) = A\sin(kx - \omega t)$, with $\omega/k = c$.

$c = \sqrt{\tau/\lambda_0}$, $K(x,t) = (\lambda_0/2)(\partial\eta/\partial t)^2$, $V(x,t) = (\tau/2)(\partial\eta/\partial x)^2$, $E(x,t) = K(x,t) + V(x,t)$, $P(x,t) = -\tau(\partial\eta/\partial t)(\partial\eta/\partial x)$ $g_x(x,t) = -\lambda_0(\partial\eta/\partial t)(\partial\eta/\partial x)$.

Traveling waves, $\eta(x,t) = f(x \pm ct)$ have special properties: $\partial\eta/\partial x = \pm(1/c)(\partial\eta/\partial t)$, $K(x,t) = V(x,t)$, $E(x,t) = \tau(\partial\eta/\partial x)^2$.

Sound: $c = \sqrt{B/\rho} = 340$ m/s for air. $p = -B(\partial s/\partial x)$, $\partial^2 s/\partial t^2 = -(1/\rho)\partial p/\partial x$.

General Solutions, bounded at infinity.
Finite string: $\eta(x,t) = \sum_{n=1}^{\infty}\sin(k_n x)\left(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right)$, $\omega_n = ck_n$, $k_n = n\pi/L$.
Infinite string: $\eta(x,t) = f(x-ct) + g(x+ct)$; initial displacement $f(x) + g(x)$, initial velocity $c(g'(x) - f'(x))$.

Boundary Conditions. Fixed: $\eta = \partial\eta/\partial t = 0$. Free: $\partial\eta/\partial x = 0$. Both fixed or free: $\omega_n = cn\pi/L$; mixed $\omega_n = c(2n+1)\pi/2L$.

Symmetries. The wave equation is symmetric under (a) Reflection along x, $\tilde{\eta}(x,t) = \eta(-x,t)$, (b) Time reversal, $\tilde{\eta}(x,t) = \eta(x,-t)$, (c) Reflection along y, $\tilde{\eta}(x,t) = -\eta(x,t)$. It is also (d) Homogeneous, $\tilde{\eta}(x,t) = \eta(x-\Delta,t)$, (e) Time independent, $\tilde{\eta}(x,t) = \eta(x,t-\Delta)$, and invariant under (f) Sideways motion, $\tilde{\eta}(x,t) = \eta(x,t) + \Delta$.

Reflection and Transmission (Massless Knot). $R = (Z_1 - Z_2)/(Z_1 + Z_2)$ and $T = 2Z_1/(Z_1+Z_2)$ where impedances $Z_i = \sqrt{\lambda_i\tau_i}$. If the tension is constant, $R = (c_2 - c_1)/(c_2 + c_1)$ and $T = 2c_2/(c_2 + c_1)$

Group and Phase Velocities. Phase velocity $\omega(k)/k$. Group velocity $d\omega/dk$. One dimensional chain $\omega(k) = \sqrt{K/M}\sqrt{2 - 2\cos(ka)}$.

Physics 218—Exam II (Wednesday November 6) Fall 2002
James P. Sethna

NAME:______________________________

The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

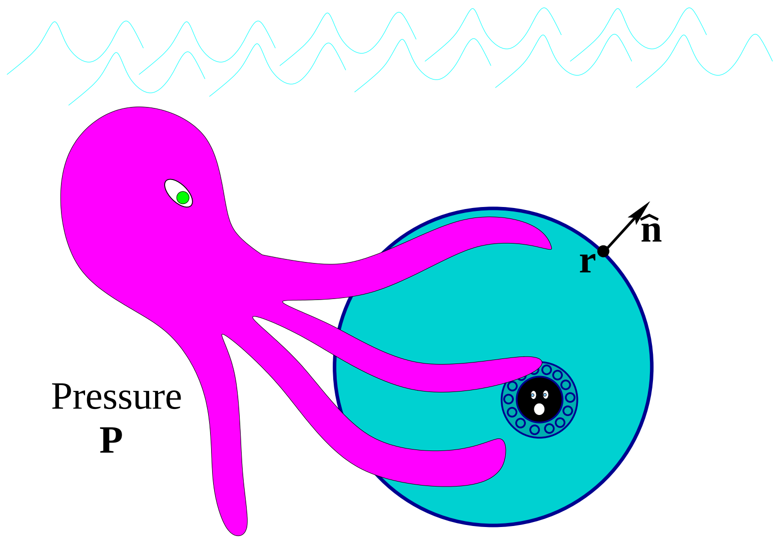
Multiple Choice (40 pts):____ **x 10 pts =** ______________

Short Answer: S1 (60 pts) ______________

TOTAL ______________

Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (10 pts) The Squid and the Bathysphere.



A scientist is exploring the bottom of the sea in a bathysphere, where the weight of the water above exerts a large, uniform pressure P. A giant squid attacks! It squeezes the diving vessel in an irregular fashion. The normal vector to a point $\mathbf{r}$ on the surface of the bathysphere is $\hat{\mathbf{n}}$. Assuming that the squeezing is slow (so that the material is in local equilibrium), and assuming that the squid is not directly pressing on the point $\mathbf{r}$, what can you say about the stress field $\sigma_{ij}(\mathbf{r})$?

(A) $\sigma_{ij}\hat{n}_i = 0$

(B) $\sigma_{ij}\hat{n}_j = 0$

(C) $\sigma_{ij}\hat{n}_j = -P\hat{n}_i$

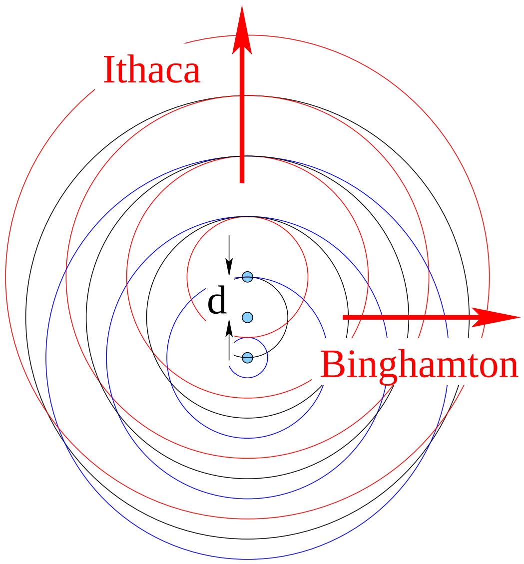
(D) $\sigma_{ij}\hat{n}_j = -P\hat{n}_j$

(E) $\mu\sigma_{ij}\sigma_{ij} + (\lambda/2)\sigma_{ii}^2 = P$

Answer

$\boxed{}$

M2. (10 pts) Antenna Array.



A radio station south of Ithaca and west of Binghamton transmits with a carrier wave of wavelength λ. They use three antennas spaced equally along a north-south direction, separated by a distance $d = 2/3\,\lambda$. The antennas are close together compared to the distance to either city. The radio station cleverly delays the signal in the antennas, so as to add a phase shift $\phi = 2\pi/3$ in the carrier wave between neighboring antennas. In particular, if we number the three antennas from south to north as $\{-1, 0, 1\}$, the radio signal is given by

$$A(r_0)\left(\exp[i(\omega t - kr_1 + \phi)] + \exp[i(\omega t - kr_0)] + \exp[i(\omega t - kr_{-1} - \phi)]\right)$$

What is the ratio of the average intensity I_{av} of the radio signal in Ithaca, compared to I_0, which one would find with only one antenna transmitting? In Binghamton?

(A) Ithaca: $I_{\mathrm{av}}/I_0 = 9$; Binghamton $I_{\mathrm{av}}/I_0 = 3$.

(B) Ithaca: $I_{\mathrm{av}}/I_0 = 9$; Binghamton $I_{\mathrm{av}}/I_0 = 1$.

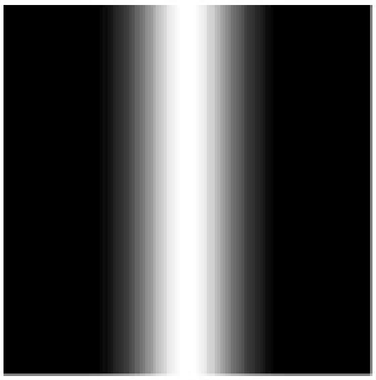
(C) Ithaca: $I_{\mathrm{av}}/I_0 = 9$; Binghamton $I_{\mathrm{av}}/I_0 = 0$.

(D) Ithaca: $I_{\mathrm{av}}/I_0 = 3$; Binghamton $I_{\mathrm{av}}/I_0 = 1$.

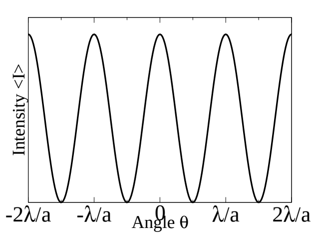
(E) Ithaca: $I_{\mathrm{av}}/I_0 = 3$; Binghamton $I_{\mathrm{av}}/I_0 = 0$.

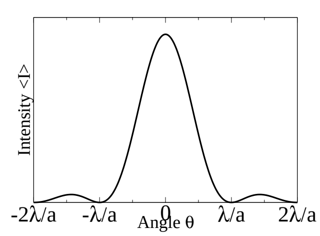
Answer

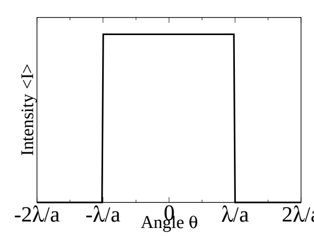
M3. (10 pts) Diffraction and Fourier Optics.

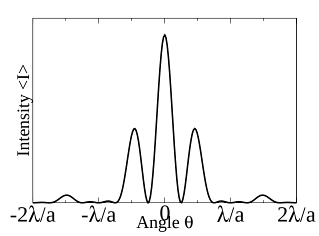


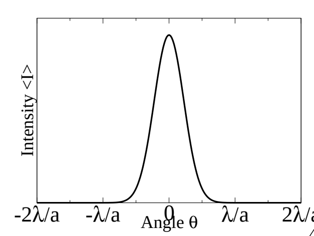
A glass slide is coated with soot, blocking all light except for a thin stripe across the center where the soot has been rubbed off. A laser beam of wavelength λ is aimed at the glass: the beam width is large compared to the slit width. A careful measurement of the light transmission immediately outside the glass shows that amplitude of the light has a Gaussian profile: it varies with x as $\exp(-2x^2/a^2)$, where a is a width of the transparent strip and x is the distance from the center of the strip. Which of the following intensity patterns will be observed on a distant screen? (You may assume small angles.)

(A)


(B)


(C)


(D)


(E)


Answer

M4. (10 pts) Tensors.

In the derivation which gave the wave equation for longitudinal sound waves, we proved the identity

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}.$$

Which of the following formulas is the same equality, but in modern tensor notation? (You don't need to *derive* the formula, only *translate* it.)

(A) $\partial_i \partial_j u_k = \partial_i \epsilon_{ijk} \partial_j u_k - \partial_i^2 u_j$

(B) $\epsilon_{ijk} \partial_i \epsilon_{jk\ell} \partial_\ell u_m = \partial_m \partial_m u_j - \partial_m \partial_j u_m$

(C) $\epsilon_{ijk} \partial_j \epsilon_{k\ell m} \partial_\ell u_m = \partial_i \partial_m u_m - \partial_j \partial_j u_i.$

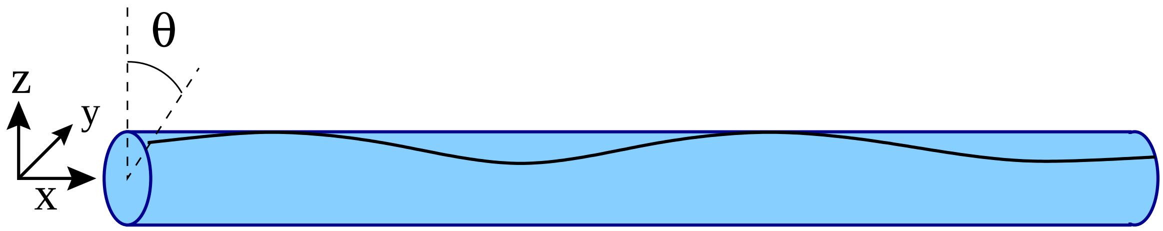
(D) $\partial_i \epsilon_{jk\ell} \partial_m \epsilon_{nop} = \partial_i \partial_j u_k - \partial_i^2 u_j$

(E) $\partial_i \epsilon_{ijk} \partial_j u_k = \partial_m \partial_m u_j - \partial_m \partial_j u_m$

Answer

Short Answer: Show Your Work

S1. (60 pts) Torsional Wave Equation.



A long thin rod is twisted about its long axis by an angle* $\theta(x) = A\cos(kx - \omega t)$, where x measures the distance along the rod and θ measures the angle with respect to the $\hat{z}$ axis, so $\mathbf{u}(x, y, z, t) = (0, y(\cos\theta - 1) - z\sin\theta, z(\cos\theta - 1) + y\sin\theta) \approx (0, -z\theta, y\theta)$ where the last approximation is true for small wave amplitudes A. In this problem you may assume that A is small, so in particular

$$\mathbf{u}(x, y, z, t) = (0, -zA\cos(kx - \omega t), yA\cos(kx - \omega t))\,.$$

(a) Write the strain tensor field, $\epsilon_{ij}(x, y, z, t)$, out as a 3×3 matrix. Ignore the geometric nonlinearity, so $\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$.

$$\epsilon_{ij}(x, y, z, t) = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

* You may use a complex wave $\theta(x) = Ae^{i(kx-\omega t)}$ if you prefer.

(b) The torsional rod is made of an isotropic material, with elastic constants λ and μ. From your answer for part (a), compute the stress tensor $\sigma_{ij}(x, y, z, t) = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$.

$$\sigma_{ij}(x, y, z, t) = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

(c) Calculate the force $F_i = \partial_j \sigma_{ij}\, dV$ on a small volume dV of material at x, y, z.

$$\mathbf{F}(x, y, z, t) = \Bigg(\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Bigg)\, dV$$

(d) Is the displacement field

$$\mathbf{u}(x, y, z, t) = (0, -zA\cos(kx - \omega t), yA\cos(kx - \omega t)) .$$

curl free or divergence free? Should the wave speed $c = \omega/k$ be the longitudinal speed of sound $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ or the transverse speed of sound $c_T = \sqrt{\mu/\rho}$?

Circle one of each (possibly one on each side).

Curl Free | Divergence Free

$c_L = \sqrt{(\lambda + 2\mu)/\rho}$ | $c_T = \sqrt{\mu/\rho}$

(e) Give the formulas for the force $\mathbf{F}$ (transcribed from part (c)), the mass m, and the accelleration $\mathbf{a}$ for a small chunk of rod of volume dV. The torsional rod has mass ρ per unit volume. The accelleration should be calculated directly from the displacement field $\mathbf{u}(x, y, z, t) = (0, -zA\cos(kx - \omega t), yA\cos(kx - \omega t))$. Show that $\mathbf{u}$ satisfies Newton's law $\mathbf{F} = m\mathbf{a}$, and solve for the dispersion relation $\omega(k)$. (Show your work: using the answer from part (d) will not suffice.)

$$\mathbf{F} = \left(\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \right)$$

$$m =$$

$$\mathbf{a} = \left(\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \right)$$

$\omega(k) =$ ________________________.

Physics 218—Exam II (November 6) Fall 2002

Formula Sheet

James P. Sethna

Sound Waves in Three Dimensions.

$\rho\partial^2\mathbf{u}/\partial t^2 = -\nabla p$, $p = -B\nabla\cdot\mathbf{u}$, $\partial^2 p/\partial t^2 = c^2\nabla^2 p$ with $c = \sqrt{B/\rho}$, $\partial^2\mathbf{u}/\partial t^2 = c^2\nabla^2\mathbf{u}$.
Spherical waves: $p(\mathbf{r}, t) = f(|\mathbf{r}| - ct)/|\mathbf{r}|$.
Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where the index of refraction $n = \sqrt{\epsilon\mu}$ is c/v.
Phase shift π on reflection where the index of refraction increases (*e.g.*, light off glass).
Intensity along the direction of propagation $I = p\partial\mathbf{u}/\partial t$,
energy density $E = (\rho/2)(\partial u/\partial t)^2 + p^2/(2B)$.
If $p(t) = \sum_n \tilde{p}_n \exp(i\omega_n t)$ and $\rho(t) = \sum_m \tilde{\rho}_m \exp(i\omega_m t)$, then the total power is the sum of the power in each frequency channel: $\sum_n (-i\omega/2)\tilde{p}_n\tilde{\rho}_n^*$.

Interference and Diffraction.

Double Slit. Phase difference $\phi = 2\pi d\sin(\theta)/\lambda = kd\sin(\theta)$.
Intensity $I_{av} = 4I_0\cos^2(\phi/2) = 4I_0\cos^2(kd\sin\theta/2)$ (I_0 single slit intensity).
Constructive for $d\sin\theta = 0, \pm\lambda, \pm2\lambda, \ldots$, destructive for $d\sin\theta = \pm\lambda/2, \pm3\lambda/2, \ldots$.
Multiple slits. $I_{av} = I_0\sin^2(N\phi/2)/\sin^2(\phi/2)$; principle maxima at $\phi = 0, 2\pi, 4\pi$, destructive at $\phi = 2m\pi/N$ with m any integer *except* $0, \pm N, \pm 2N, \ldots.$
Diffraction. If the slit opening is $f(x)$, $I_{av} \propto |\tilde{f}(k\sin\theta)|^2$.
The Fourier transform of a shifted function $f(x-\Delta)$ is $\exp(-i\Delta k)\tilde{f}(k)$.
Single wide slit. $I_{av} = I_{center}\sin^2\alpha/\alpha^2$ with $\alpha = ak\sin(\theta)/2$.

Tensor Notation.

Einstein convention: $a_{ijk\ell}b_{imno} = \sum_{i=1}^{3} a_{ijk\ell}b_{imno}$.
Dot product $\mathbf{a}\cdot\mathbf{b} = a_i b_i$, matrix applied to vector $(M\mathbf{x})_i = M_{ij}x_j$, matrix multiplication $(MN)_{ij} = M_{ik}N_{kj}$, trace $Tr(M) = M_{ii}$.
Laplacian $\nabla^2 f = \partial_i\partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$, divergence $\nabla\cdot\mathbf{v} = \partial_i v_i$.
Identity tensor δ_{ij}, equals one if $i = j$, zero otherwise.
Totally antisymmetric tensor $\epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kij}$. $\epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312}$, $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$, zero if any index repeats.
$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k$, $(\nabla\times\mathbf{v})_i = \epsilon_{ijk}\partial_j v_k$, $\det M = \epsilon_{ijk}\epsilon_{\ell mn}M_{i\ell}M_{jm}M_{kn}$.
$\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

Elasticity Theory.

Stress tensor $\sigma_{ij}\hat{\mathbf{n}}_j$ = Force/Area across surface perpendicular to $\hat{\mathbf{n}}$. $\sigma_{ij} = \sigma_{ji}$ because torques on small volumes must vanish. Force on a small volume $F_i = \partial_j\sigma_{ij}$. For hydrostatic pressure P, $\sigma_{ij} = -P\delta_{ij}$.
Strain tensor $\varepsilon_{ij} = (1/2)\left(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k\right)$, where the last term (the geometric nonlinearity) is usually ignored. $\varepsilon_{ij} = \varepsilon_{ji}$. The strain tensor for uniform stretching $-\Delta V/V = 3\Delta L/L$ would be $\varepsilon_{ij} = (\Delta L/L)\delta_{ij}$.

Tensor of elasticity $c_{ijk\ell}$ gives Hooke's law for anisotropic media, $\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}$. $c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$. There are 21 possible independent elastic constants.
The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}$.
Isotropic moduli. The bulk modulus K is the same as B for fluids: $P = -K(\Delta V/V)$. Under a shear by an angle θ, $E = (1/2)\mu\theta^2$.
Under unconstrained stretching, $F/A = Y\Delta L/L$, and $\Delta W/W = -\nu\Delta L/L$, where ν is Poisson's ratio. $K = 2\mu/3 + \lambda$, $\nu = \lambda/2(\mu+\lambda)$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu+\lambda)$.
Isotropic Tensors. $c_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.
Wave equations. $\rho_0\partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijk\ell}\partial_j(\partial_k u_\ell + \partial_\ell u_k)$. For isotropic media, $\rho_0\partial^2 u_i/\partial t^2 = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$, or $\rho_0\partial^2\mathbf{u}/\partial t^2 = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$.
Decomposing $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$ with $\nabla\cdot\mathbf{u}_T = 0$ and $\nabla\times\mathbf{u}_L = 0$, we have $\partial^2\mathbf{u}_T/\partial t^2 = c_T^2\nabla^2\mathbf{u}_T$ and $\partial^2\mathbf{u}_L/\partial t^2 = c_L^2\nabla^2\mathbf{u}_L$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda+2\mu)/\rho}$.

WKB.

If $c(x)$ varies slowly, the solution to a wave equation $\partial^2\eta/\partial t^2 = c(x)^2\partial^2\eta/\partial x^2$ is approximately of the WKB form $\eta(x,t) = A(x)\cos\left((S(x) - \omega t)\right)$, where the phase $dS(x)/dx = \omega/c(x) = k(x)$ and the amplitude $A(x) = A(x_0)\sqrt{c(x)/c(x_0)}$.

Formulas from Prelim I.

Trigonometry $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.
Wave Equation Solutions. The wave equation

$$\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\exp\left(i(kx - \omega t)\right)$, where $\omega/k = c$.
Fourier Transform of a Gaussian. If $f(x) = (1/\sqrt{2\pi}\sigma)\exp\left(-x^2/2\sigma^2\right)$, $\tilde{f}(k) = \exp\left(-\sigma^2 k^2/2\right)$.
)

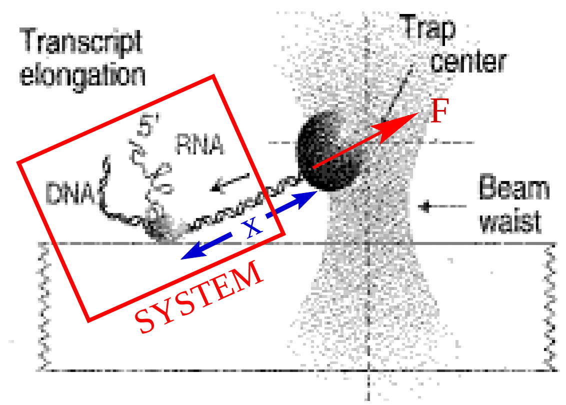
Physics 218—Final Exam (December 18) **Fall 2002**
James P. Sethna

NAME:______________________________

M1 (25 pts)	____________
M2 (30 pts)	____________
M3 (30 pts)	____________
S1 (30 pts)	____________
S2 (30 pts)	____________
S3 (20 pts)	____________
S4 (35 pts)	____________
TOTAL (200 pts)	____________

Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (25 pts) Molecular Motors: Which Free Energy?



An RNA polymerase molecular motor attached to a glass slide is pulling along a DNA molecule (transcribing it into RNA). The opposite end of the DNA molecule is attached to a bead which is being pulled by an optical trap with a constant external force F. Let the distance from the motor to the bead be x: thus the motor is trying to move to decrease x and the force is trying to increase x.

This is a complicated example! It is under two constant forces (F and pressure), and involves complicated chemistry and biology. Nonetheless, you know some things based on fundamental principles. Let us consider the optical trap and the distant fluid as being part of the external environment, and define the "system" as the local region of DNA, the RNA, motor, and the fluid and local molecules in a region immediately enclosing the region, as shown above. Without knowing anything further about the chemistry or biology in the system, which two of the following must be true on average, in all cases, according to basic laws of thermodynamics?

(T) (F) The total entropy of the universe (the system, bead, trap, laser beam ...) must increase with time.

(T) (F) The entropy S_s of the system must increase with time.

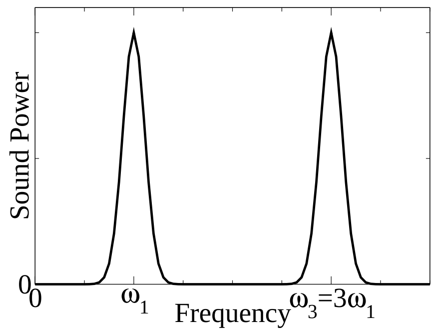
(T) (F) The total energy E_T of the universe must decrease with time.

(T) (F) The energy E_s of the system must decrease with time.

(T) (F) $G_s - Fx = E_s - TS_s + PV_s - Fx$ must decrease with time, where G_s is the Gibbs free energy of the system.

Note: F is a force, not the Helmholtz free energy. Precisely two of the answers are correct.
Related formulæ: $G = E - TS + PV$.

M2. (30 pts) Wave Frequencies on a String.



At left is the sound power from a microphone measuring sound from a vibrating string. Assume the emitted power at a given frequency is proportional to the energy density in that mode.

(A) (10 points) $\eta(x,t)$ is clearly a superposition of two standing waves, of amplitudes A_1 and A_3 for frequencies ω_1 and ω_3. What is the ratio of the amplitudes $|A_3/A_1|$, given that the energy density in the two channels is the same (from the figure above)?

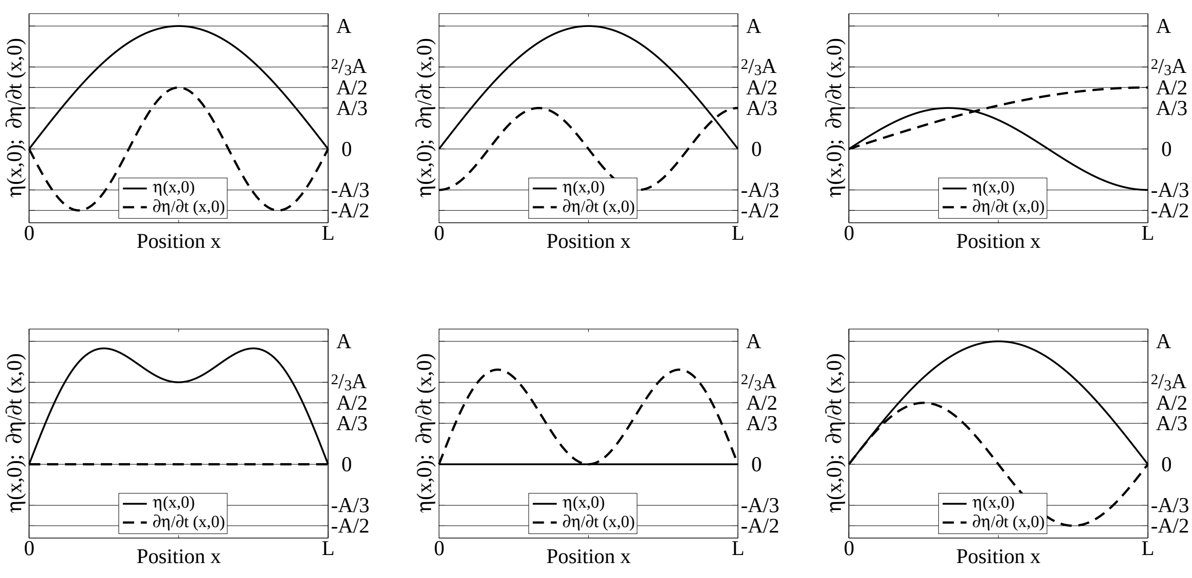
(a) $|A_3/A_1| = 1$.
(b) $|A_3/A_1| = 1/3$.
(c) $|A_3/A_1| = 3$.
(d) $|A_3/A_1| = 1/9$.
(e) $|A_3/A_1| = 9$.

Answer

Related formulæ: Standing wave $\eta(x,t) = A\sin(kx)\cos(\omega t)$ (for example).
$U = \int_0^L (1/2)\tau(\partial\eta/\partial x)^2 + (1/2)\lambda_0(\partial\eta/\partial t)^2\, dx$; standing $= (1/4)\tau k^2 A^2 L = (1/4)\lambda_0^2\omega^2 A^2 L$.

The figures below show candidate initial conditions $\eta(x,0)$ (solid) and velocities $\partial\eta/\partial t\ (x,t)$ (dashed) for waves on strings of length L.

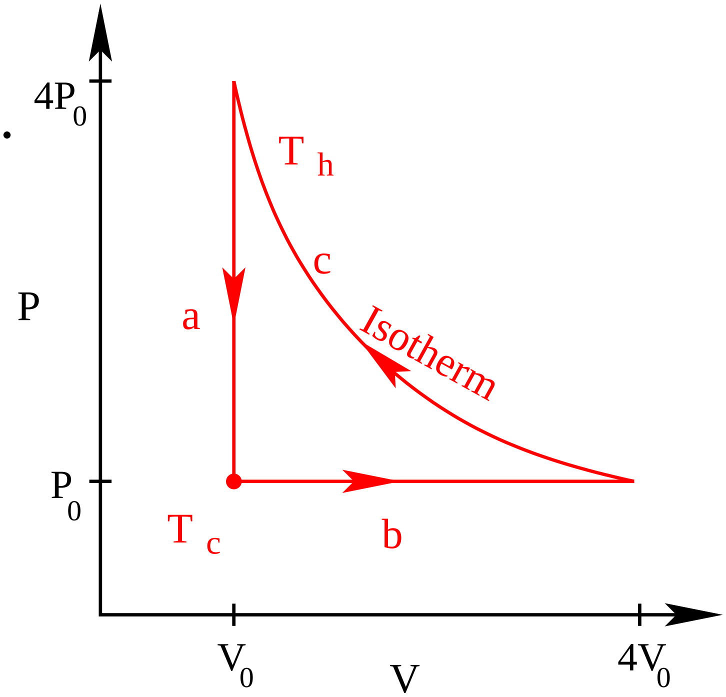
(B) (20 points) There are a variety of ways to get the power spectrum shown in part (A). Cross out the two figures below which are NOT possible initial conditions, because they either will evolve with the wrong frequencies, or are not compatible with energy-conserving boundary conditions (fixed, free, or mixed). [Checking the amplitudes will take a lot of time: you can rule out the wrong ones using wavelengths and nodal positions alone.]



Note: precisely two should be crossed out. If you want to check amplitudes, you'll need to know the fundamental frequency ω_1 in each case is 0.5 radians per second.

Related formulæ: Standing wave $k_n = n\pi/L$ (fixed, free); $k_n = (2n-1)\pi/2L$ (mixed). $\omega = ck = \sqrt{\tau/\lambda_0}\,k$.

M3. (30 pts) P-V Diagrams.



A monatomic ideal gas in a piston is cycled around the path in the P-V diagram shown. Leg **a** cools at constant volume by connecting to a heat bath at T_c; leg **b** heats at constant pressure by connecting to a heat bath at T_h; leg **c** compresses at constant temperature while remaining connected to the bath at T_h. Which of the following are true? (5 points each.)

(T) (F) The cycle is reversible: no net entropy is created in the universe.

(T) (F) The cycle acts as a refrigerator, using work from the piston to draw energy from the cold bath into the hot bath, cooling the cold bath.

(T) (F) The cycle acts as an engine, transferring heat from the hot bath to the cold bath and doing positive net work on the outside world.

(T) (F) The work done per cycle has magnitude $|W| = P_0 V_0 |4 \log 4 - 3|$.

(T) (F) The heat transferred into the cold bath, Q_c has magnitude $|Q_c| = (9/2) P_0 V_0$.

(T) (F) The heat transferred from the hot bath Q_h, plus the net work W done by the piston onto the gas, equals the heat Q_c transferred into the cold bath.

Related formulæ: $PV = Nk_BT$, $U = (3/2)Nk_BT$, $\Delta S = Q/T$, $W = -\int P dV$, $\Delta U = Q + W$. Notice that the signs of the various terms depend on convention (heat flow out vs. heat flow in): you should figure the signs on physical grounds.

Short Answer: Show Your Work

S1. (30 pts) Photon diffusion in the Sun.

Most of the fusion energy generated by the Sun is produced near its center. The Sun is 7×10^5 km in radius. Convection probably dominates heat transport in approximately the outer third of the Sun, but it is believed that energy is transported through the inner portions (say to a radius $R = 5 \times 10^8$ m) through a random walk of X-ray photons. (A photon is a quantized package of energy: you may view it as a particle which always moves at the speed of light c. Ignore for this problem the index of refraction of the Sun.) Assume that the mean free path ℓ for the photon is $\ell = 5 \times 10^{-5} m$. About how many random steps N will the photon take of length ℓ to get to the radius R where convection becomes important? About how many years Δt will it take for the photon to get there? (You may assume for this problem that the photon takes steps in random directions, each of equal length given by the mean-free path.)

$N =$ ______________________

$\Delta t =$ ______________________ years

Related formulæ: $c = 3 \times 10^8$ m/s; $\langle x^2 \rangle \sim 2Dt$; $\langle s_n^2 \rangle = n\sigma^2 = n\langle s_1^2 \rangle$.

There are $31,556,925.9747 \sim \pi \times 10^7 \sim 3 \times 10^7$ seconds in a year.

S2. (30 pts) Two–state system.

Consider the statistical mechanics of a tiny object with only two discrete states:* one of energy E_1 and the other of higher energy $E_2 > E_1$.

(S2.A) (15 pts) **Boltzmann probability ratio.** Find the ratio of the equilibrium probabilities ρ_2/ρ_1 to find our system in the two states, when weakly coupled to a heat bath of temperature T. What is the limiting probability as $T \to \infty$? As $T \to 0$?

$\rho_2(T)/\rho_1(T) =$ ______________________

$\rho_2(0)/\rho_1(0) =$ ______________________

$\rho_2(\infty)/\rho_1(\infty) =$ ______________________

Related formulæ: Boltzmann probability $= Z(T) \exp(-E/kT) \propto \exp(-E/kT)$.

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* Visualize this as a tiny biased coin, which can be in the 'heads' or 'tails' state but has no other internal vibrations or center of mass degrees of freedom. Many systems are well described by large numbers of these two–state systems: some paramagnets, carbon monoxide on surfaces, glasses at low temperatures, ...

(S2.B) (15 pts) **Probabilities and averages.** Use the normalization of the probability distribution (the system must be in one or the other state) to find ρ_1 and ρ_2 separately. (That is, solve for $Z(T)$ in the 'related formula' for part (A).) What is the average value of the energy E?

$\rho_1(T) =$ ________________________

$\rho_2(T) =$ ________________________

$\langle E \rangle =$ ____

S3. (20 pts) Elastic Deformation Tensors.

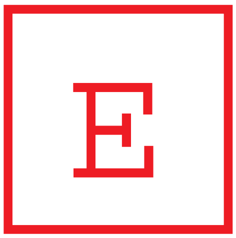
A cube is deformed in the xy plane according to the mapping $(x, y, z) \to (x + y, y, z)$.

(S3.A) (10 pts) **Elastic tensor.** Find the displacement field $\mathbf{u}(x, y, z)$ and the elastic strain tensor ε (including the geometric nonlinearity).

$$\mathbf{u}(x, y, z) = \left(\qquad , \qquad , \qquad \right)$$

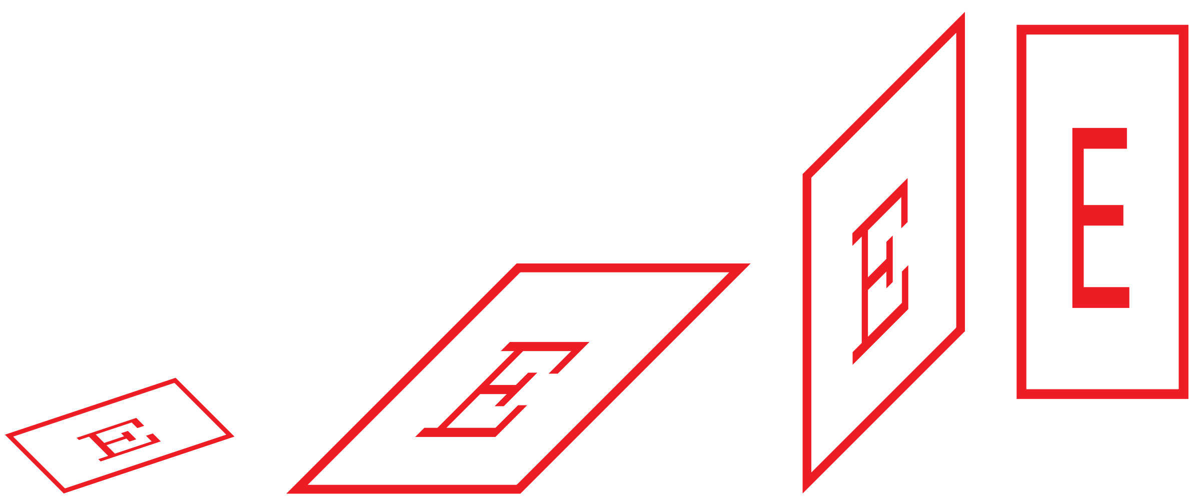
$$\varepsilon = \begin{pmatrix} \quad , & \quad , & \quad \\ \quad , & \quad , & \quad \\ \quad , & \quad , & \quad \end{pmatrix}$$

Related formulæ: $\varepsilon_{ij} = (1/2)\left(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k\right)$.

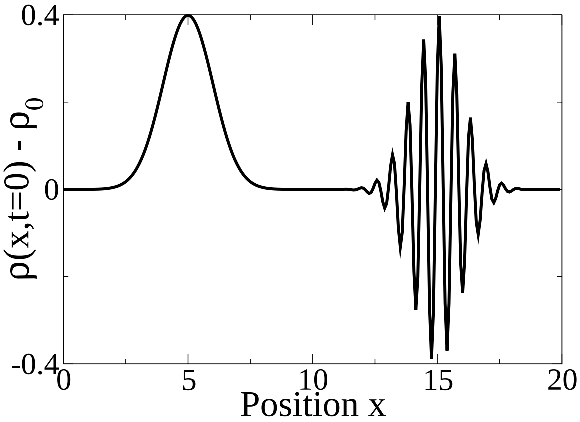


The view of the undistorted cube from the top (along the z axis) is shown at left. We showed on the homework that if you include the geometric nonlinearity, rotating a body gave zero strain. More generally, it's true that rotating an already deformed body doesn't change the strain tensor.

(S3.B) (10 pts) **Pictures.** The pictures below are views from the top of a variety of cubes: the cubes are only deformed and rotated in the xy plane. Circle the two pictures that have the strain tensor requested above.



S4. (35 pts) Solving Diffusion: Fourier and Green.



An initial density profile $\rho(x, t = 0)$ is perturbed slightly away from a uniform density ρ_0, as shown at left. The density obeys the diffusion equation $\partial\rho/\partial t = D\partial^2\rho/\partial x^2$, where $D = 0.001\ \mathrm{m}^2/\mathrm{s}$. The lump centered at $x = 5$ is a Gaussian $\exp(-x^2/2)/\sqrt{2\pi}$, and the wiggle centered at $x = 15$ is a smooth envelope function multiplying $\cos(10x)$.

(S4.A) (10 pts) **Fourier.** As a first step in guessing how the pictured density will evolve, let's consider just a cosine wave. If the initial wave were $\rho_{\cos}(x, 0) = \cos(10x)$, what would it be at $t = 10\mathrm{s}$?

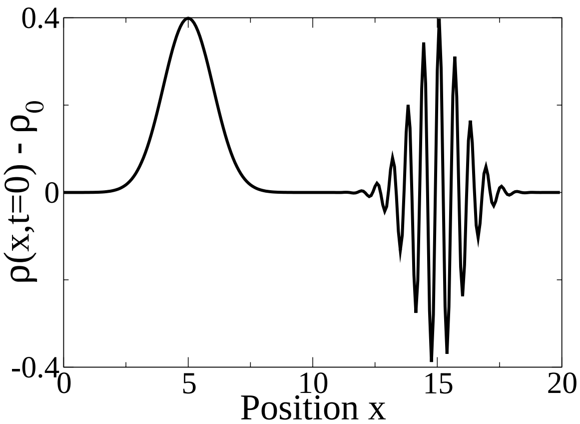
$\rho_{\cos}(x, 10) =$ ____.

Related formulæ: $\tilde{\rho}(k, t) = \tilde{\rho}(k, t')\tilde{G}(k, t - t')$; $\tilde{G}(k, t) = \exp(-Dk^2 t)$.

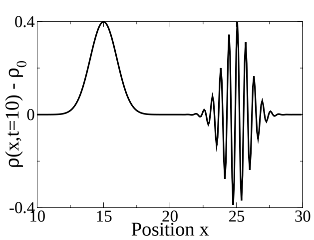
(S4.B) (15 pts) **Green.**
As a second step, let's check how long it would take to spread out as far as the Gaussian on the left. If the wave at some earlier time $-t_0$ were a δ function at $x = 0$, $\rho(x, -t_0) = \delta(x)$, what choice of the time elapsed t_0 would yield a Gaussian $\rho(x, 0) = \exp(-x^2/2)/\sqrt{2\pi}$ for the given diffusion constant $D = 0.001\mathrm{m}^2/\mathrm{s}$?

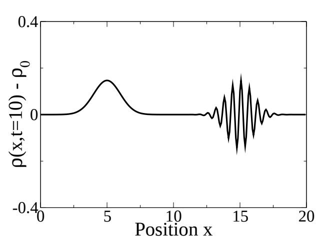
$t_0 =$ __________________________.

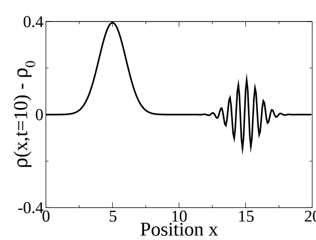
Related formulæ: $\rho(x, t) = \int \rho(y, t')G(y - x, t - t')\, dy$;
$G(x, t) = (1/\sqrt{4\pi Dt}) \exp(-x^2/(4Dt))$.

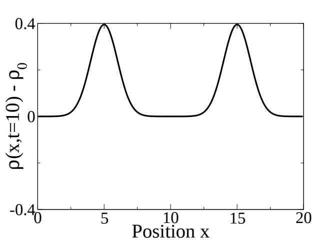


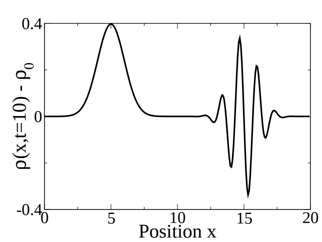
(S4.C) (10 pts) **Pictures.** Now consider time evolution for the next ten seconds. The initial density profile $\rho(x, t = 0)$ is again shown at left. Which of the following represents the density at $t = 10$s? (Hint: compare $t = 10$s to the time t_0 from part (B).)

(A) 

(B) 

(C) 

(D) 

(E) 

Answer

Related formulæ: $\langle x^2 \rangle \sim 2Dt$;