Physics 218: Waves and Thermodynamics
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Fill Out Before Fourier Lab
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Preparation for Fourier Lab

Complete these exercises before coming to Fourier lab, (Monday evening 9/22 and Thursday afternoon 9/23, Rockefeller B3, hidden around the corner in the basement.) You'll need to show the lab TA the comparison between your predictions and what you found in the lab. These should be quick, since you've covered this material thoroughly in your homeworks.

Definitions: Fourier Series, Fourier Transforms, and FFTs.

The Fourier series for periodic functions of period L is

$$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx, \qquad (FS1)$$

where $k_m = 2\pi m/L$. The Fourier series can be resummed to retrieve the original function:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x). \qquad (FS2)$$

The Fourier transform for functions on the infinite interval is

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x)\exp(-ikx)\,dx \qquad (FT1)$$

where now k takes on all values. We regain the original function by doing the inverse Fourier transform.

$$y(x) = (1/2\pi)\int_{-\infty}^{\infty} \tilde{y}(k)\exp(ikx)\,dk \qquad (FT2),$$

The Fast Fourier transform starts with N equally spaced data points y_ℓ, and returns a new set of complex numbers $\tilde{y}_m^{FFT}$:

$$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N), \qquad (FFT1)$$

with $m = 0, \ldots N-1$. It's essentially sampling the function $y(x)$ at equally spaced points points $x_\ell = \ell L/N$ for $\ell = 0, \ldots N-1$.

$$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-ik_m x_\ell), \qquad (FFT2)$$

I. Sinusoidal Waves and Fourier Series

In the first part of the lab, we will take the Fourier series of periodic functions $y(x) = y(x + L)$ with $L = 20$. We will sample the function at $N = 32$ points, and using a FFT to approximate the Fourier series and Fourier transform. The Fourier series and transform will be plotted as functions of k, at $-k_{N/2}, \ldots, k_{N/2-2}, k_{N/2-1}$. (Remember from problem set 4 that the negative m points are given by the last half of the FFT.)

(a) Find the Fourier series $\tilde{y}_m$ in this interval for $\cos(k_1 x)$ and $\sin(k_1 x)$. (Hint: they are zero except at two values of $m = \pm 1$.

$\cos(k_1 x)$: $\tilde{y}_{\pm 1} =$

$\sin(k_1 x)$: $\tilde{y}_{\pm 1} =$

(b) What spacing δk between k-points k_m do you expect to find? What is $k_{N/2}$? Evaluate each as a formula and numerically (*i.e.* your first answer will be $2\pi/L = \pi/10 = 0.31416$).

$\delta k =$

$k_{N/2} =$

The Fourier series $\tilde{y}_m$ runs over all integers m. The fast Fourier transform runs only over $0 \leq m < N$. There are three ways to understand this difference: function space dimension, wavelengths, and aliasing.

Function space dimension. The space of periodic functions $y(x)$ on $0 \leq x < L$ is infinite, but we are sampling them only at $N = 32$ points. The space of possible fast Fourier series must also have N dimensions! Now, each coefficient of the FFT is complex (two dimensions), but the negative frequencies are complex conjugate to their positive partners (giving two net dimensions for the two wavevectors k_m and $k_{-m} \equiv k_{N-m}$). If you're fussy, $\tilde{y}_0$ has no partner, but is real (only one dimension), and if N is even $\tilde{y}_{-N/2}$ also is partnerless, but is real. So N k-points are generated by N real points!

Wavelength. The points that we sample the function are spaced $\delta x = L/N$ apart. It makes sense that the fast Fourier transform would stop when the wavelength becomes close to δx: how can we resolve wiggles shorter than our sample spacing?

(c) Give a formula for y_ℓ in equation (FFT1) for a cosine wave at k_N, the first wavelength *not* calculated by our FFT. It should simplify to a constant. Give the simplified formula for y_ℓ at $k_{N/2}$, the first missing wavevector after we've shifted the large m's to $N - m$ to get the negative frequencies.

$y_\ell = y(x_\ell) = \cos(k_N x_\ell) =$

$y_\ell = y(x_\ell) = \cos(k_{N/2} x_\ell) =$

So, the FFT returns Fourier components only until there is one point per bump (half-period) in the cosine wave.

Aliasing. Suppose our function really does have wiggles with shorter distances than our sampling distance δx. Then it's fast Fourier transform will have contributions to the long-wavelength coefficients $\tilde{y}_m^{FFT}$ from these shorter wavelength wiggles: specifically $\tilde{y}_{m\pm N}$, $\tilde{y}_{m\pm 2N}$, etc.

(d) Let's work out a simple case of this: a short-wavelength cosine wave. You showed in problem 4.6(b) that, on our sampled points x_ℓ, $\exp ik_{m\pm N} x_\ell = \exp ik_m x_\ell$. Show that the short wavelength wave $\cos(k_{m+N} x_\ell) = \cos(k_m x_\ell)$, and hence that its fast Fourier transform will have bogus contributions at the long wavelength k_m.

II. Gaussian Packets and Fourier Transforms

In the second part of the lab, we will consider the Fourier transforms of certain functions defined on an infinite interval. The most important of these is the Gaussian, which you worked with in problem set 3:

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-(x-x_0)^2/2\sigma^2\right). \tag{3.5.3}$$

We'll study how the Fourier transform $\tilde{G}(k) = \exp(-ikx_0)\exp(-\sigma^2 k^2/2)$ changes as we change the width σ and the center x_0.

Notice that when we make the Gaussian narrower (smaller σ) its Fourier transform gets *wider*. Shorter lengths mean higher frequencies.

(a) Show that this is true in general. Change variables in equation (FT1) above to show that if $z(x) = y(Ax)$, that $\tilde{z}(k) = \tilde{y}(k/A)/A$.

Notice that when we move the center of our function x_0, the Fourier transform gets multiplied by a phase $\exp(-ikx_0) = \cos(kx_0) - i\sin(kx_0)$.

(b) Show that this is true in general: change variables in equation (FT1) above to show that if $z(x) = y(x - x_0)$ that $\tilde{z}(k) = \exp(-ikx_0)\tilde{y}(k)$. How does this change the power spectrum $|\tilde{z}(k)|^2$?