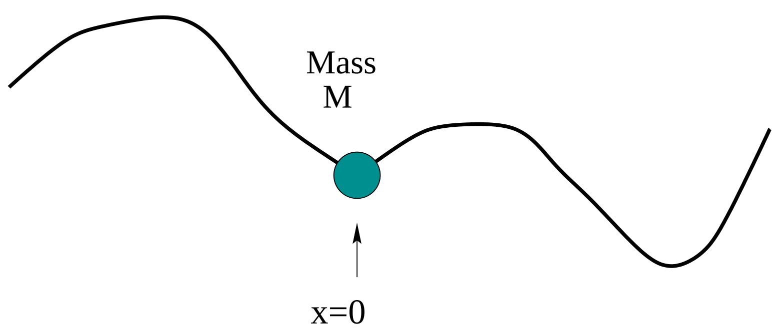
Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 1, due Monday Sept. 1
Latest revision: August 28, 2003, 8:46 pm

Reading

Elmore & Heald, section 1.1
Feynman, I.22-5/6

Problems

(1.1) Bead on a String.



A bead of mass M is attached at $x = 0$ to a string of mass density λ_0 stretched with horizontal tension τ. The string extending to the left (negative x) has height $\eta_1(x,t)$, and to the right it has height $\eta_2(x,t)$. The bead has height $y(t) = \eta_1(0,t) = \eta_2(0,t)$. What is the formula for the acceleration of the bead? (Draw the free body diagram!)

(A) $d^2y/dt^2 = (\tau/M)\partial^2\eta/\partial x^2$

(B) $d^2y/dt^2 = (\tau/M)(\eta_2(0,t) - \eta_1(0,t))$

(C) $d^2y/dt^2 = (\tau/M)(\eta_1(0,t) - \eta_2(0,t))$

(D) $d^2y/dt^2 = (\tau/M)\left(\frac{\partial\eta_2}{\partial x}|_{x=0} - \frac{\partial\eta_1}{\partial x}|_{x=0}\right)$

(E) $d^2y/dt^2 = (\lambda_0/2\tau)\left(\frac{\partial^2\eta_1}{\partial x^2}|_{x=0^-} + \frac{\partial^2\eta_2}{\partial x^2}|_{x=0^+}\right)$

(1.2) Fourier Series. The laws for the motion of stretched strings, of the surface of water, of sound, and of electromagnetic radiation are called *wave equations* because they all have special solutions of the form of sinusoidal waves. That is, a string with initial height $A\sin(kx)$ or $B\cos(kx)$ will time evolve in a particularly simple way. We need to review some mathematics about sinusoidal waves.

(a) (Review) What is the wavelength of the shape $A\sin(kx)$, where x is the distance measured along the string?

We call k the *wave vector* for the wave.

(b) (Review) Suppose we study a stretched string with the ends at $x = 0$ and $x = L$ held fixed at height $y = 0$. Calculate the values k_m at which $\eta(x) = A\sin(kx)$ satisfies these two boundary conditions. (To be specific, let $m - 1$ be the number of zeros, or nodes, for $y(x)$ inside the string, not including the boundaries. For this problem, all values of k_m should be positive.)

In this course, we will make extensive use of complex numbers. In quantum mechanics, the waves really involve complex amplitudes, but for this course the complex numbers are just a way to make the mathematics simpler: our waves will be the real parts of complex waves. You should remember the formula

$$\exp(ikx) = \cos(kx) + i\sin(kx). \tag{1.2.1}$$

Thus cosine waves are the real part of the complex wave $\exp(ikx)$.

(c) (Review) If k is positive, for what smallest positive value of x_0 is the real part of $\exp(ik(x - x_0))$ a sine wave, $\sin(kx)$?

The Fourier series for a function $y(x)$ is an expansion in terms of sinusoidal waves. Elmore and Heald concentrate on the Fourier sine and cosine expansions. In our work, we'll use complex Fourier series. Suppose we have a function $y(x)$ defined on $0 \le x \le L$, with $y(0) = y(L)$. (This is called *periodic boundary conditions*, since we can make y into a periodic function by placing new copies side-by-side over each period L.) Various mathematical theorems tell us that we can write $y(x)$ as an infinite series

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x). \tag{1.2.2}$$

in terms of the complex sinusoidal waves $\exp(ik_m x)$ which satisfy the same boundary condition.

(d) Show that $y(x) = \exp(ik_m x)$ satisfies $y(0) = y(L)$ if $k_m = 2\pi m/L$ (here k_m may be positive or negative). Are these the same wave vectors as you found in part (b)?

The formula for the complex Fourier series coefficients $\tilde{y}_m$ of a function $y(x)$ in an interval of length L is

$$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx. \tag{1.2.3}$$

Mathematical theorems tell us that the sum in equation (1.2.2) converges to $y(x)$ if we use the coefficients from equation (1.2.3). Also, the coefficients are unique: if the coefficients aren't all the same, the functions are different.

(e) Use equation (1.2.3) to compute the Fourier coefficients $\tilde{y}_m$ with $m = -1$, 0, and 1, for $\sin(2\pi x/5)$, in an interval of length $L = 5$. Check this using the well-known formula $\sin(\theta) = (\exp(i\theta) - \exp(-i\theta))/2i$. Without using the formula (1.2.3), but using the fact that the coefficients are unique, give *all* the Fourier coefficients for $7\cos(18\pi x/5)$, again with $L = 5$. (Hint: what's the well-known formula for $\cos(\theta)$?)

Decomposing a function into a Fourier series, equation (1.2.2), is like writing a vector as a sum $\mathbf{v} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$. Instead of a three-dimensional space of vectors, we have an infinite-dimensional space of functions. Our "unit vectors" are the complex exponential waves $\exp(ik_m x)$. Finding the coefficients, equation (1.2.3), is like taking the dot product to find the coefficient in the expansion, $a_x = \mathbf{v}\cdot\hat{\mathbf{x}}$, etc., except that the dot product of two complex functions is generalized to an integral of one times the complex conjugate of the other,

$$f \cdot g = (1/L)\int_0^L f(x)g^*(x)\,dx. \qquad (1.2.4)$$

The dot products of different unit vectors $\hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = 0$: they are orthogonal to one another. Also, the unit vectors are normalized, so $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = 1$.

(f) Show that our Fourier series functions $\exp(ik_m x)$ with $k_m = 2\pi m/L$ are normalized under the dot product (1.2.4). Show that any two different Fourier series functions $m \neq n$ are orthogonal under the dot product (1.2.4).

(1.3) Fourier Series: Computer Lab.

Download the executables for *Fourier* from the bottom of the course home page. (The Windows version works, as far as I know. The Linux version may not: let me know if you have success or not, and which version of Linux you run.) When you start it up, you'll find at left a graph of the function $y(x)$, and at right the Fourier series $\tilde{y}(k_m)$ for the function. Thus to get m in $\tilde{y}_m$, take the coordinate along the horizontal axis and multiply by $L/2\pi$.

Use *Fourier* to check your answers for problem 1.2(e). In particular, (i) set $L = 5$, (ii) find m and x_0 to plot $y(x) = \sin(2\pi x/5)$, and (iii) read off the Fourier coefficients from the right graph. (You can zoom in with the mouse. Black is the real part, red the imaginary part.) Do they agree with those you found in 1.2(e)? Then (iv) find m, x_0, and the amplitude A to generate $7\cos(18\pi x/5)$, (v) double the number of points N repeatedly until $y(x)$ looks like a smooth sine wave, and (vi) read off the Fourier transform. Does it agree?