Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 2, due Monday Sept. 8
Latest revision: September 10, 2003, 10:14 am

Reading

Elmore & Heald, sections 1.2, 1.3, 1.4, 1.5, 1.6, 1.7
Feynman, sections I.22-5, I.22-6, I.23 Feynman, sections I.50-1/4

Problems

Elmore & Heald, page 7, problem 1.2.3 (stationary initial condition), and page 13, problem 1.3.2 (a).

Quick ones.

Elvis. Elvis notices that his A string on his guitar is off pitch: it is vibrating at 445 Hz. He wants it to sound at 440 Hz.
(a) Is his guitar string sharp (too high pitch) or flat (too low)?
(b) Elvis twists the little knob at the top of the string to tune it to 440 Hz. Did he tighten or loosen the tension?
(c) By what percentage does he change the tension?

Sympathetic Vibration. Consider two strings of equal mass density and length. When the strings are near each other, starting string 1 vibrating in its fundamental mode causes string 2 to vibrate in its third (n=3) natural mode. What is the ratio of the tension of string 1 to string 2?

Numerical Derivatives. The angle $\theta(t)$ of a pendulum is measured at three different times: $\theta(1.8) = 0.72$, $\theta(2.0) = 0.74$, and $\theta(2.2) = 0.82$. Estimate the accelleration $\partial^2\theta/\partial t^2$ at $t = 2.0$.

Big ones.

(2.1) Solving the Wave Equation Numerically

Consider a string of length L that is shaken up and down at the left end $\eta(0,t) = f(t)$ and is fixed in position $\eta(L,t) \equiv 0$ at the right end.

$$\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2} \tag{1}$$

To solve this equation numerically, we must discretize the string into chunks of size δx in space, and take small, discrete time steps δt in time.

(a) Derive the approximate formula for the second derivative

$$\frac{\partial^2 \eta}{\partial x^2} \approx \frac{\eta(x+\delta x, t) - 2\eta(x,t) + \eta(x-\delta x, t)}{\delta x^2} \tag{2}$$

from the approximate formula for the first derivative

$$\frac{\partial \eta}{\partial x}(x_0) \approx \frac{\eta(x_0+\epsilon/2) - \eta(x_0-\epsilon/2)}{\epsilon}. \tag{3}$$

(Hint: pick $\epsilon = \delta x$ and $x_0 = x \pm \delta x/2$. It may help to draw a picture of where you are evaluating the first and second derivatives.)

(b) Applying this approximate formula to the wave equation (1), show that we can write the future position of the string in terms of the past and present. If our wire is broken up into N chunks of size $\delta x = L/N$,

$$x_0 \equiv 0, \quad x_1 = \delta x, \quad \ldots \quad x_N = N\delta x \equiv L \tag{4}$$

show that

$$\eta(x_i, t+\delta t) \approx 2\eta(x_i, t) - \eta(x_i, t-\delta t) + (c\,\delta t/\delta x)^2 \left(\eta(x_{i+1}, t) - 2\eta(x_i, t) + \eta(x_{i-1}, t)\right). \tag{5}$$

Notice that this equation applies for $i = 1, \ldots N-1$, but not for $i = 0$ or $i = N$. These *boundary conditions* have to be supplied separately: in our case, fixed on the right, forced on the left.

(c) Write a program (using Matlab, Mathematica, a spreadsheet, or any other method of your choice) to solve this wave equation with $L = 15m$, $c = 2m/s$, $\delta x = 0.5m$, $\delta t = 0.1s$, and

$$f(t) = \exp(-(6-t)^2/4). \tag{6}$$

Use the evolution equation (5) and the initial conditions

$$\eta(x_i, 0) \equiv \eta(x_i, -\delta t) \equiv 0. \tag{6}$$

When should the pulse center hit the right end of the string? Plot the pulse shape when the center is partway to the wall, when your analysis says it should be hitting the wall (notice the numerical error in our calculation), and after it is reflected. Where do you think the energy is stored when the pulse is at the wall?

(2.2) Fourier Series and Gibbs Phenomenon

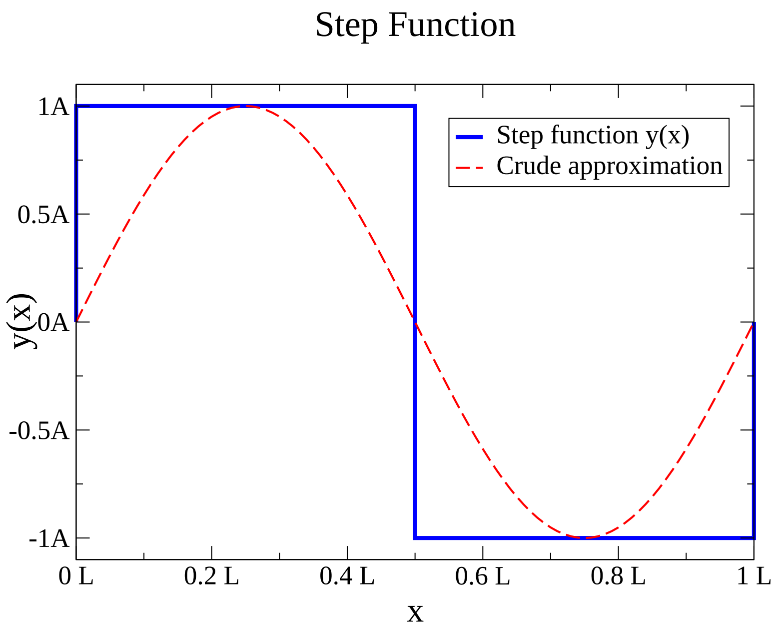

Figure 2.2.1 Step Function.

We defined complex Fourier series in the last problem set:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x), \tag{1.2.2}$$

$$\tilde{y}_m = (1/L) \int_0^L y(x) \exp(-ik_m x) dx, \tag{1.2.3}$$

with $k_m = 2\pi m/L$. In this problem set, we'll look at the Fourier series for a couple of simple functions, the step function (above) and the triangle function.

Consider a function $y(x)$ which is A in the range $0 < x < L/2$ and minus A in the range $L/2 < x < L$ (shown above). It's a kind of step function, since it takes a step downward at $L/2$.*

(a) As a crude approximation, the step function looks a bit like a chunky version of a sine wave, $A \sin(2\pi x/L)$. In this crude approximation, what would the complex Fourier series be?

(b) Show that the odd coefficients for the complex Fourier series of the step function are $\tilde{y}_m = -2Ai/(m\pi)$ (m odd). What are the even ones? Check that the coefficients $\tilde{y}_m$ with $m = \pm 1$ are close to those you guessed in part (a).

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* It can be written in terms of the standard Heaviside step function $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$, as $y(x) = A\,(1 - 2\Theta(x - L/2))$.

(c) Setting $A = 2$ and $L = 10$, plot the partial sum of the series equation (1.2.2) for $m = -n, -n+1, \ldots, n$ with $n = 1$, 3, and 5. (You'll likely need to combine the coefficients $\tilde{y}_m$ and $\tilde{y}_{-m}$ into sines or cosines, unless your plotting package knows about complex exponentials.) Does it converge to the step function? If it is not too inconvenient, plot the partial sum up to $n = 100$, and concentrate especially on the overshoot near the jumps in the function at 0, $L/2$, and L. This overshoot is called the Gibbs phenomenon, and occurs when you try to approximate functions $y(x)$ with discontinuities.

One of the great features of the Fourier series is that it makes taking derivatives and integrals easy.

(d) Show that the Fourier series of the derivative of a function $y'(x) = dy/dx$ is $\tilde{y'}_m = ik_m\tilde{y}_m$. Show, for $m \neq 0$, that the Fourier series for the integral of a function $y(x)$ is $\tilde{y}_m/(ik_m)$.

What does the integral of our step function look like? Let's sum the Fourier series for it!

(e) Consider the Fourier series whose coefficients are $\tilde{y}_m/(ik_m)$, where $\tilde{y}_m$ is the complex Fourier series you defined in part (b), and where you can set the $m = 0$ coefficient to zero. This series should sum to an integral of the step function. Do partial sums up to $\pm m = n$, with $n = 1$, 3, and 5, again with $A = 2$ and $L = 10$. Would the derivative of this function look like the step function? If it's convenient, do $n = 100$, and notice there are no overshoots.