**Physics 218: Waves and Thermodynamics**
Fall 2003, James P. Sethna
**Homework 3, due Monday Sept. 15**
Latest revision: September 3, 2003, 10:08 pm

## Reading

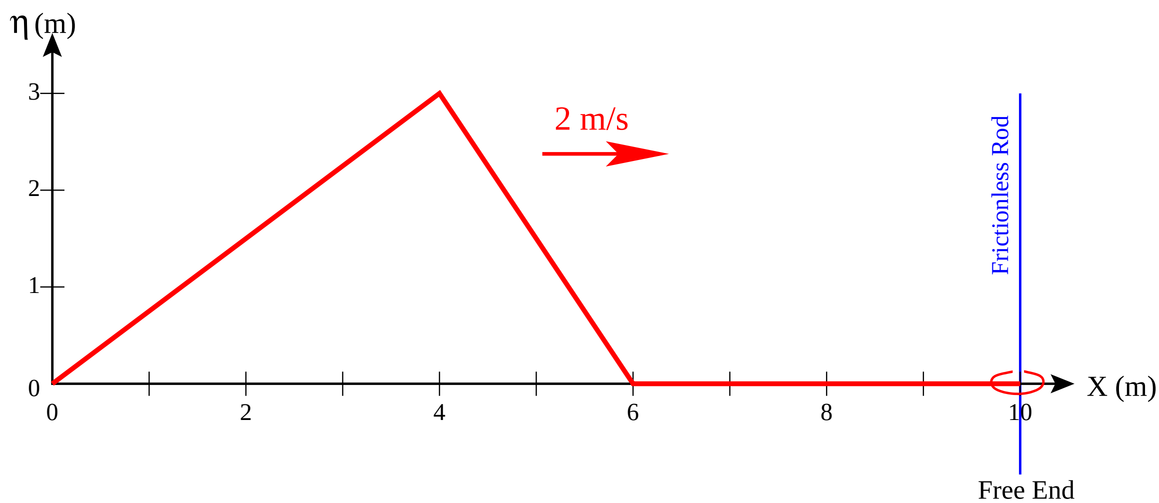
Elmore & Heald, sections 1.6, 1.7, 1.8, 1.9

Feynman, I.23, I.49-1/2, I.50-1/4

## Problems

Elmore & Heald, page 38, problems 1.8.1 (Steel wire), 1.8.4 (Continuity equation for energy density).

**(3.1) Traveling Wave on a String.** The figure below shows a traveling wave propagating to the right on a string at time $t = 0$. The tension is $8N$ and the string has mass per unit length $2kg/m$. The string has length $10m$ and has a fixed end at $x = 0$ and a free end at $x = 10m$.




(a) Draw a graph of the transverse velocity (chunk velocity) of the wave at time $t = 0$, labeling your axes and giving units.
(b) Draw graphs of the energy density, the power, and the momentum density of the wave at $t = 0$.
(c) Draw graphs of the height of the wave and its transverse velocity at $t = 4$ seconds. Show that the total energy is the same as that at $t = 0$. Is the total momentum the same?
(d) Draw a graph of the transverse velocity at $x = 5$ as a function of time, from $t = -1$ second to $t = 4$ seconds.
(e) A new pulse of the same shape but twice as high and half as wide is sent down the wire. The energy density plot will be half as wide (why?) and how many times as tall? How much will the total energy change?

**(3.2) Fourier wave.** A musical instrument playing a note of frequency $\omega_1$ generates a pressure wave $P(t)$ periodic with period $2\pi/\omega_1$: $P(t) = P(t+2\pi/\omega_1)$. The complex Fourier series of this wave is zero except for $n = \pm 1$ and $\pm 2$, corresponding to the fundamental $\omega_1$ and the first overtone. At $n = 1$, the Fourier amplitude is $2 - i$, at $n = -1$ it is $2 + i$, and at $n = \pm 2$ it is 3. What is the pressure $P(t)$?

(A) $\exp((2+i)\omega_1 t) + 2\exp(3\omega_1 t)$

(B) $\exp((2\omega_1 t))\exp(i(\omega_1 t)) * 2\exp(3\omega_1 t)$

(C) $\cos 2\omega_1 t - \sin\omega_1 t + 2\cos 3\omega_1 t$

(D) $4\cos\omega_1 t - 2\sin\omega_1 t + 6\cos 2\omega_1 t$

(E) $4\cos\omega_1 t + 2\sin\omega_1 t + 6\cos 2\omega_1 t$

**(3.3) Pythag: Resonance.**

We'll be using a few computer simulations to illustrate ideas from the course. We don't expect long writeups. Download the program pythag, from the course Web site (or directly from links at the bottom of

http://www.physics.cornell.edu/sethna/teaching/sss/pythag/pythag.htm).

The download will contain several programs: look for pythag.exe.

Play with the program for a while. Observe the effects of fixed, free, and reflectionless boundary conditions. Using fixed boundary conditions on both sides, and "Wave" forcing on the left, hit "Initialize" and "Run": the system is periodically forced on the left boundary at a frequency $\Omega$ and with an amplitude $A$ that you can set on the Configure menu. Change $\Omega$ to 10 rad/s, $A$ to 0.01, and the time to run on the main controls to 10 s. (You need to hit Enter to get changes to register: the number turns red to warn you.) Notice that the string wiggles under the external forcing, but the amplitude never gets very large.

Now, using the tension $\tau$, the mass per unit length $\mu_1$ (what Elmore & Heald calls $\lambda_0$), and a length $L$ (all given under the Configure menu), find the frequencies $\omega_m$ of the standing waves. Change the frequency of the forcing frequency $\Omega$ to the frequency $\omega_1$ of the fundamental mode, and reduce $A$ to 0.002. How does the amplitude in the fundamental mode build up? The small graph on the lower left shows the height $Y$ of the center of the string (our $\eta$) as a function of time: it should be oscillating with an increasing amplitude $\eta_{max} \sim t^\zeta$ as the resonance builds up. Do the peaks seem to be growing linearly in time ($\zeta = 1$), or quadratically ($\zeta = 2$), or what?

In your writeup, we'd like to see the frequency that you forced the program to excite the fundamental, and a brief, qualitative description of the growth of the oscillation peaks in time.

**(3.4) Pythag: Energy and Power.**

Let a pulse be traveling down the string at the velocity of sound $\eta(x, y) = f(x - vt)$. Use the fact that this is a traveling wave to derive a formula giving the ratio of the potential energy density to the kinetic energy density. Restart pythag (or select DEFAULT on the presets), and verify your formula. (For the writeup, just note the maximum amplitude for the kinetic and potential energy densities $KE$ and $PE$.)

Derive a formula relating the power and the total energy density $u$ for a traveling wave. Verify your formula with pythag.

Be sure to remember: these two formulas only apply to traveling waves!

**(3.5) Fourier Transforms**

In problem set 1, we defined the complex Fourier series of a function confined to an interval $(0, L)$. Waves on strings, rods, and in boxes and tanks are all confined to defined regions, but many waves are unconfined. Fourier transforms are like Fourier series, except that the range of the function goes from $(-\infty, \infty)$. The Fourier transform of a function $y(x)$ is another function $\tilde{y}(k)$:

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x) \exp(-ikx)\, dx \tag{3.5.1}$$

and you can retrieve the original function back by using the inverse Fourier transform:

$$y(x) = (1/2\pi) \int_{-\infty}^{\infty} \tilde{y}(k) \exp(ikx)\, dk \qquad (3.5.2).$$

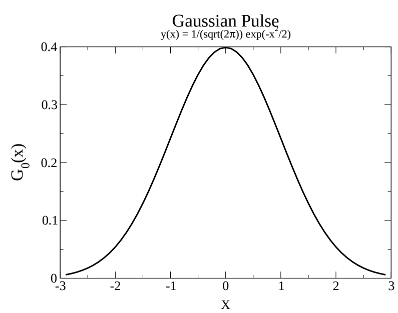



**Figure 3.5.1** Gaussian Pulse centered at $x_0 = 0$ of width $\sigma = 1$.

The famous function

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x - x_0)^2/2\sigma^2) \qquad 3.5.3$$

is usually called a normal distribution or a normalized Gaussian. It peaks at $x_0$, and as $x - x_0 \to \pm\infty$ the Gaussian dies rapidly to zero (because of the exponential of minus $x^2$). In fact, it starts getting small at about $|x - x_0| = \pm\sigma$. Thus the function is a pulse of width $\sigma$ centered at $x_0$. It is of fundamental important in probability theory, in quantum mechanics, and in statistical mechanics (last month of this course). It is also a good example of a pulse (like the sound you might get from slapping your hand on the table). Let's call $G_0(x)$ the Gaussian with mean $x_0$=0 and width $\sigma = 1$, pictured above.

(a) Show that the Fourier transform $\tilde{G}(k) = \exp(-ikx_0)\tilde{G}_0(\sigma k)$, by changing variables in equation (3.5.1) from $x$ to $z = (x - x_0)/\sigma$. Notice that you should not need to do any integrals!

The Gaussian $G(x)$ has some nice properties: the integral (norm) $\int_{-\infty}^{\infty} G(x)\, dx = 1$, the mean $\int_{-\infty}^{\infty} xG(x)\, dx = x_0$, the variance (or square of the width) $\int_{-\infty}^{\infty} (x-x_0)^2 G(x)\, dx = \sigma^2$. Also, the Fourier transform of the standard Gaussian $G_0(x)$ of width one and mean zero $\tilde{G}_0(k) = \exp(-k^2/2)$. The derivation for three of these four formulas is a bit tricky, so treat them as given.

(b) Using the formulas above and your answer for part (a), give the general formula for the real and imaginary parts of $\tilde{G}(k)$. Draw pictures of the answer for $\sigma = 2$ and $x_0 = 4$, going from $k = -2.5$ to $k = 2.5$.

The Fourier transform of a Gaussian centered at zero is another Gaussian! It's not normalized, though: its height is always one at $k = 0$.

(c) In general, the value of the Fourier transform $\tilde{y}(k)$ at $k = 0$ gives which basic property of $y$, the norm, mean, or variance?