Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 4, due Monday Sept. 22
Latest revision: September 12, 2003, 10:54 am

Experimental Lab I

Standing Waves, Monday evening 9/15 and Tuesday afternoon 9/16, Rock B26 and B30.

Computer Lab

Fourier Lab, Monday evening 9/22 and Tuesday afternoon 9/23, Rock B3 (hidden around the corner in the basement).

Reading

Elmore & Heald, sections 4.1, 4.7, 12.1, 12.2

Feynman I.47, I.48-1/4, I.50-1/4

Problems

Elmore & Heald, page 41, problems 1.9.2 (Bead on a String). Use Pythag to see that the pulse indeed does not stay the same shape: (a) set reflectionless boundary conditions on both sides, (b) force with a pulse on the left, (c) make $\mu_2 = 20$, (d) make $X_{12} = 4.99$ and $X_{23} = 5.01$, (e) set the graph time step to one and the amplitude A to 0.1.
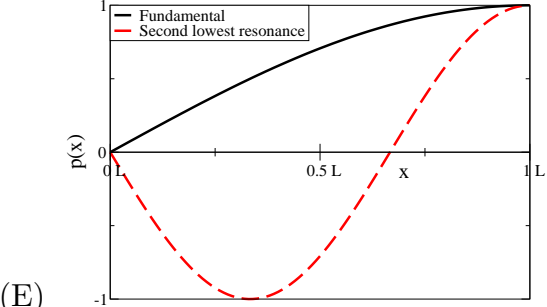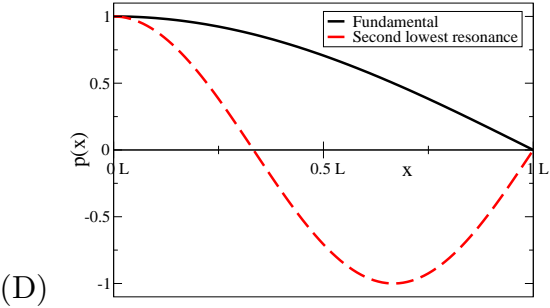
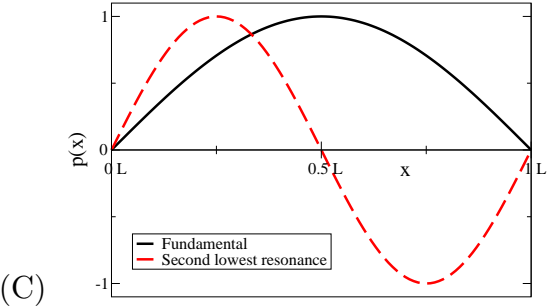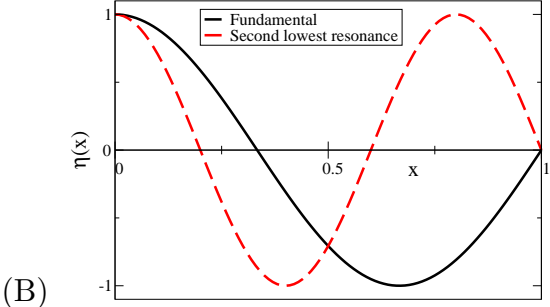**(4.1) Reflection and Transmission.**
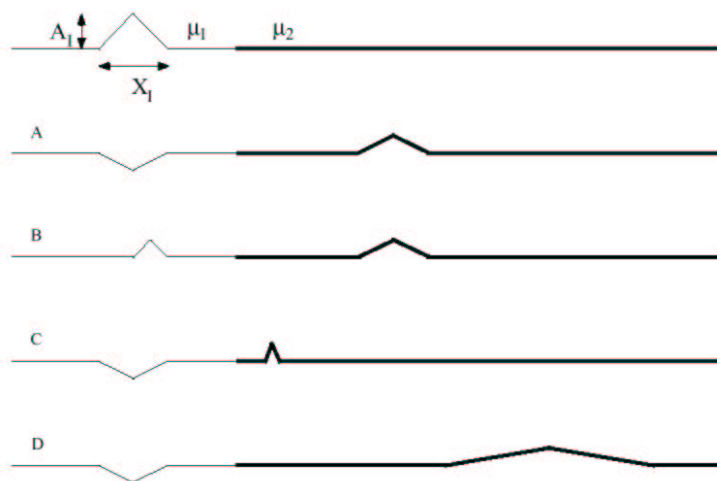
A pulse of height A_I, width X_I, travels on a string of mass density μ_1 and is incident on a string of mass density $\mu_2 = 9\mu_1$. The strings are joined together, and have the same tension. Which picture correctly describes the string after the pulse has interacted with the junction between the two strings? The pictures are drawn to scale.

(4.2) Air in a tube.
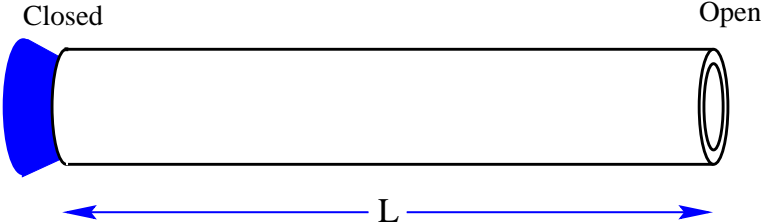
A tube of air of length L is closed on the left-hand side and open on the right. Which pictures represent the *pressure* change $p(x)$ from atmospheric pressure for the lowest two resonant frequencies in this tube? (The solid line is the fundamental, the dashed line represents the second lowest tone.)

(A)

(B)

(C)

(D)

(E)

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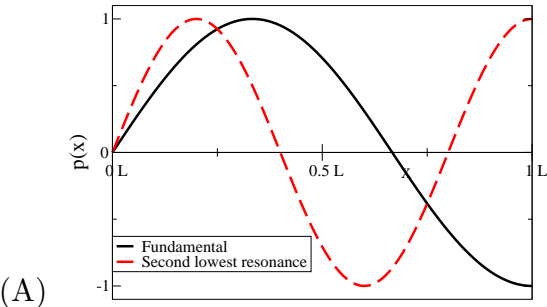
(4.3) Pythag: Reflection and Transmission.

Start up pythag, and choose the PRESET for STEPDOWN. The string comes in two pieces, whose mass densities μ_1 and μ_2 can be read off the Configure menu. The thickness of the lines roughly corresponds to the mass densities. To repeat a run, first Initialize, then Run.

Notice some qualitative facts. (1) The pulses leaves the simulation without reflection at the boundaries. I had to carefully match impedences at the boundary to avoid reflections. (2) Notice that the string is continuous and has a continuous derivative at the junction. (You can slow the pulse by lowering "graph time skip" on the Configure menu.) (3) How do the widths of the reflected and transmitted pulses compare to the incident pulse? How about their duration in time, passing by a particular place? Why should their durations agree? (4) Which is largest, the incident, reflected, or transmitted pulse? According to your transmission formula, should that always be the case if $\mu_2 < \mu_1$? (5) Does the reflected pulse invert or not? How about the reflected pulse for STEPUP? By taking the limits where the mass density ratio goes to zero and infinity, argue why this is related to reflection at fixed and free boundary conditions.

(4.4) Atoms: Dispersion and the 1-D Crystal.

In lecture we derived the equation of motion for the longitudinal displacements u_n of the nth atom in a chain of atoms connected by springs,

$$\frac{\partial^2 u_n}{\partial t^2} = (K/M)[u_{n+1} - 2u_n + u_{n-1}] \tag{4.4.1}$$

where K is the spring constant and M is the atomic mass. Assume a plane-wave solution

$$u_n = \sin(kna - \omega t) \tag{4.4.2}$$

where a is the equilibrium distance between atoms.

(a) **Dispersion Relation.** Plug in the trial solution equation (4.4.2) into equation (4.4.1). Rewrite $u_{n\pm1}$ by expanding the sines, $\sin(k(n\pm1)a - \omega t) = \sin((kna - \omega t) \pm ka) = \sin(kna - \omega t)\cos(ka) \pm \cos(kna - \omega t)\sin(ka)$ and hence write your equation in the form $-\omega^2 (BLAH) = f(k)(BLAH)$. Solve for the *dispersion relation*, the frequency $\omega(k) = \sqrt{-f(k)}$ for each wave-vector k in our one-dimensional crystal.

(b) **Continuum Limit.** What is the speed of sound for our chain at long wavelengths? To be specific, what is $\omega(k)/k$ (the phase velocity) as the wavelength goes to infinity and hence $k \to 0$? (A Taylor series under the square root might be useful.)

In the regular wave equation, where $\omega(k) = ck$, both the group velocity $d\omega/dk$ and the phase velocity $\omega(k)/k$ give the speed of sound, independent of k.

(c) Using $K/m = a = 1$, plot the dispersion relation $\omega(k)$ for $-\pi/a < k < \pi/a$. On a second graph, plot the group velocity and the phase velocity for $0 < k < \pi/a$. Which velocity is larger, for this dispersion relation?

(4.5) Decibels. The power difference between sound A and B in dB is $10 \log_{10}(P_A/P_B)$, where P is the power (energy per unit area). (Bels are named after Alexander Graham Bell, the telephone guy; a decibel is a tenth of a Bel.) The threshold of hearing is around zero decibels (0 dB). The threshold of pain is about 120 dB, and corresponds to a power of about $1 W/m^2$. From this and your knowledge of air and sound, estimate the amplitude of the vibration of your eardrum at the threshold of audibility. (The bulk modulus of air B is about $1.4 \times 10^5 \, N/m^2$; the density of air is about $1.2 \, kg/m^3$; the speed of sound in air is about $340 \, m/s$; a typical sound frequency might be $1000 \, Hz$.) Compare this with other natural scales of length: which is it closest to, the size of your ear, the width of a hair in your cochlea, the width of a cell, the width of an atom, ...

(4.6) Fourier Series, Fourier Transforms, and FFTs.
In problem set 1, we introduced the Fourier series for periodic functions of period L,

$$\tilde{y}_m = (1/L) \int_0^L y(x) \exp(-ik_m x)dx, \tag{1.2.3}$$

where $k_m = 2\pi m/L$. The Fourier series, we saw explicitly in problem set 2, can be resummed to retrieve the original function:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x). \tag{1.2.2}$$

In problem set 3, we introduced the Fourier transform for functions on the infinite interval

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x) \exp(-ikx) \, dx \tag{3.5.1}$$

where now k takes on all values. We regain the original function by doing the inverse Fourier transform.

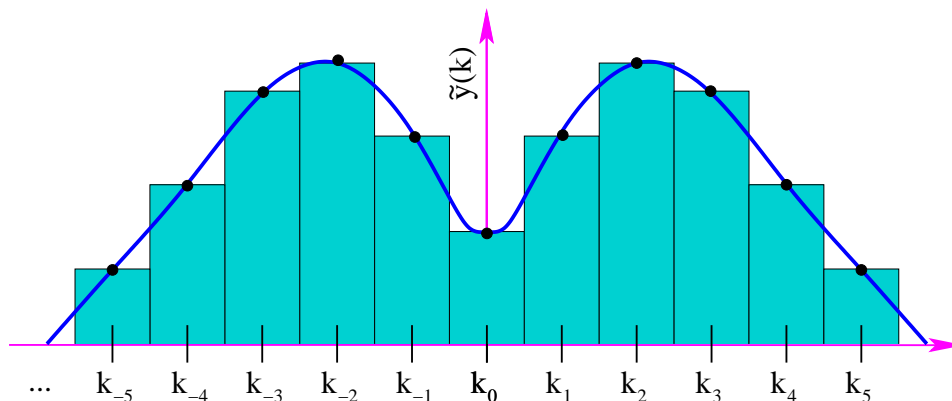$$y(x) = (1/2\pi) \int_{-\infty}^{\infty} \tilde{y}(k) \exp(ikx) \, dk \tag{3.5.2},$$

Figure 4.6, Approximating the integral as a sum. By approximating the integral $\int \tilde{y}(k) \exp(-ikx) \, dk$ as a sum over the equally spaced points k_m, $\sum_m \tilde{y}(k) \exp(-ik_m x)\Delta k$, we can connect the formula for the Fourier transform to the formula for the Fourier series.

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(a) **Series → Transform.** Let $y(x)$ be a smooth function which is zero outside $(0, L)$. By what constant do you need to multiply the Fourier series coefficient $\tilde{y}_m$ in equation (1.2.3) to get the Fourier transform $\tilde{y}(k_m)$ in (3.5.1)? Approximating the Fourier transform integral (3.5.1) as a sum (as shown in Figure 4.6), use the Fourier series formulas (1.2.2) and (1.2.3) to explain or derive the factor $(1/2\pi)$ in equation (3.5.2).

As we take $L \to \infty$ the spacing between the points k_m, $2\pi/L$, gets smaller and smaller, and the approximation of the integral as a sum gets better and better.

There is a remarkably fast numerical method, called the Fast Fourier transform. It starts with N equally spaced data points y_ℓ, and returns a new set of complex numbers $\tilde{y}_m^{FFT}$:

$$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N). \qquad (4.6.1)$$

(b) **FFT → Series.** We can use the FFT to give an approximation to the Fourier series. Let $y_\ell = y(x_\ell)$ where $x_\ell = \ell(L/N) = \ell(\Delta x)$. As in part (a), approximate the Fourier series integral (1.2.3) above as sum over y_ℓ, $(1/L) \sum_{\ell=0}^{N-1} y(x_\ell) \exp(-ik_m x_\ell)\Delta x$. For small positive m, give the constant relating $\tilde{y}_m^{FFT}$ to the Fourier series coefficient $\tilde{y}_m$. The Fourier series is defined for both positive and negative m, where the FFT gives only positive m. Show that $\tilde{y}_m^{FFT} = \tilde{y}_{m+N}^{FFT}$, and then argue how you can use this to get the Fourier series coefficients for negative m by looking at the FFT near the end of the list,