

Physics 218: Waves and Thermodynamics

Fall 2003, James P. Sethna

Homework 5, due Monday Sept. 28

Latest revision: October 2, 2003, 4:45 pm

Computer Labs

Fourier Lab I, Monday evening 9/22 and Tuesday afternoon 9/23, Rock B3 (hidden around the corner in the basement).

Prelim I

Prelim I is scheduled next week, Friday October 3. The content will focus on the homework, the experimental lab *Standing Waves* and the two Fourier labs. There will be multiple-choice questions (no partial credit) and one or perhaps two longer multiple-part essay questions. Next Monday, instead of a problem set, I will pass out copies of last year's Prelim I for you to use while studying.

Reading

Elmore & Heald, sections 5.1-5.3, 5.10, 12.3/5

Feynman, section I.48-5/6, I.49-3/5, I.50-5/6, I.51 1/2, I.52, II.25-10.

Problems

Elmore & Heald, page 142, problem 5.2.3 (squeaky voices with Helium).

(5.1) Deriving the Wave Equation: Symmetries *vs.* Free Body Diagram

A thin horizontal string of density λ and tension τ is vibrating inside a viscous fluid. It is subject to a transverse viscous force $b \partial \eta / \partial t$ per unit length so as to oppose the transverse motion of the string. In addition, it is subject to an external gravitational force.

(a) **Free Body Diagram Method.** Generalize the derivation of E&H equation (1.1.2) to incorporate these effects of viscosity and gravity. Make sure to draw the appropriate free body diagram for the chunk of string with two tension forces, the viscous force, and the force due to gravity.

Part (a) could have asked you to "Incorporate into the wave equation the leading order terms breaking time-reversal invariance and invariance under changing the sign of the order parameter. Give a possible physical origin for each term." You've already solved this problem in part (a): part (b) asks you to translate the answer into the language of modern condensed-matter physics:

(b) **Symmetry Method.** Which term, gravity or friction, breaks time-reversal invariance? Which term breaks invariance under changing the sign of the order parameter?

(5.2) Deriving New Laws.

The evolution of a physical system is described by a field Ξ , obeying a partial differential equation

$$\partial\Xi/\partial t = A \partial\Xi/\partial x. \quad (5.2.1)$$

(a) Symmetries.

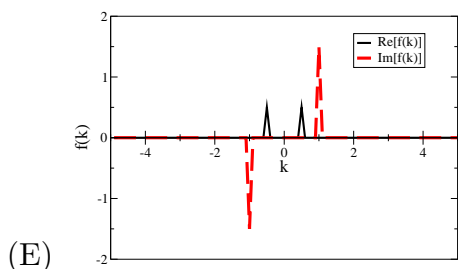
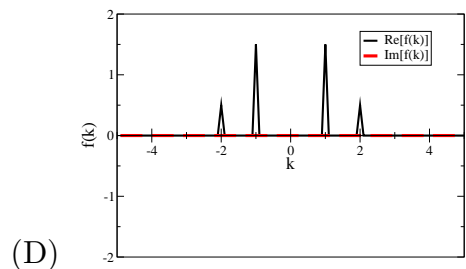
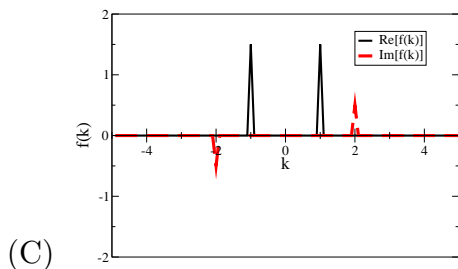
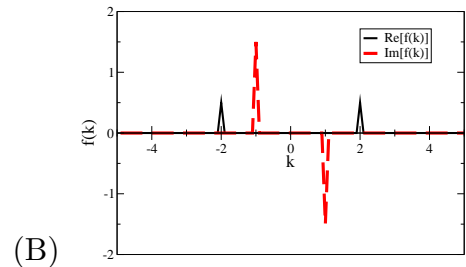
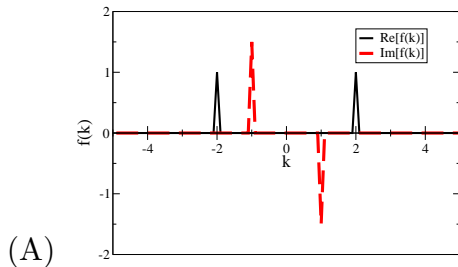
Give the letters corresponding to ALL the symmetries that this physical system appears to have:

- (A) Spatial inversion ($x \rightarrow -x$).
- (B) Time reversal symmetry ($t \rightarrow -t$).
- (C) Order parameter inversion ($\Xi \rightarrow -\Xi$).
- (D) Homogeneity in space ($x \rightarrow x + \Delta$).
- (E) Time translational invariance ($t \rightarrow t + \Delta$).
- (F) Order parameter shift invariance ($\Xi \rightarrow \Xi + \Delta$).

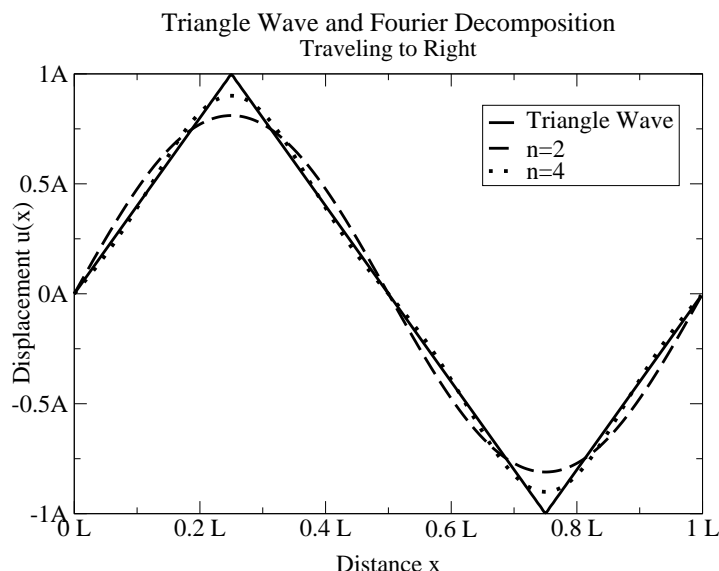
(b) **Traveling Waves.** Show that our equation $\partial\Xi/\partial t = A \partial\Xi/\partial x$ has a traveling wave solution. If $A > 0$, which directions can the waves move?

(5.3) Fourier Series.

Which picture represents the Fourier series associated with the function $f(x) = 3 \sin(x) + \cos(2x)$? (The solid line is the real part, the dashed line is the imaginary part.)



(5.4) Sawtooth Wave.



A sound wave generator generates a triangular pressure air wave moving toward the right down a hollow tube, as shown in the figure above. The triangles repeat forever with wavelength L . The maximum displacement of the wave is A , the velocity of sound is v , and the bulk modulus for air is B .

- (a) What is the intensity (power per unit area) traveling down the tube?

The figure shows the Fourier series for our wave truncated at $n = \pm 2$ and $n = \pm 4$.

- (b) We now want to decompose this intensity into different frequencies. Give the time average intensity I_n^{av} of a single traveling plane wave of wave vector k_n and amplitude a_n , $u_n(x, t) = a_n \sin(k_n(x - vt))$? (Leave your answer in terms of a_n and k_n .)

The Fourier series for the displacement of the wave is

$$u(x) = \sum_{n=0}^{\infty} a_n \sin(k_n(x - vt))$$

with $k_n = 2\pi n/L$. The Fourier coefficients are $a_n = 0$ for n even, and for n odd are

$$a_n = (-1)^{(n-1)/2} 8A/(\pi^2 n^2).$$

- (c) Verify explicitly that the sum of the intensities per frequency channel n you calculated in part (b) equals the total intensity you calculated in part (a). You'll need the formula $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \dots$

This is Feynman's energy theorem, section I.50-5: the energy of the sum of different Fourier waves is the sum of the energies of the individual waves. This is why we can talk about the power spectrum of a wave: you can think of the power at different frequencies as being independent of one another.

(5.5) Pythag: Group velocity, phase velocity, and dispersion.

Start up Pythag. Choose *Packet* forcing on the left-hand side: this yanks on the left sinusoidally with a frequency $\Omega = 300$ radians per second, with an amplitude given by a Gaussian pulse of FWHM 0.04 seconds. Hit *Initialize* and *Run*, and watch the packet bounce back and forth. As is usual with the wave equation, the pulse propagates without changing in shape. This is only true, however, so long as the pulse does not change much on the length scale given by the distance between points δx on the numerical string.

Open the *Configure* menu. Change Ω to 800 and *FWHM* to 0.015. To slow down the pulse, lower *graph time skip* to one. You should now see a pulse which changes shape as it moves.

- (a) Is the group velocity faster or slower than the phase velocity? This is easiest to see by looking at the pulse early on, before it stretches out: do the peaks within the wave of the carrier frequency move forward faster or slower than the pulse as a whole?

After several passes across the window, you should see a broad pulse, which has longer waves on one side than the other.

- (b) Does the leading edge have longer or shorter wavelength than the trailing portion of the packet? Which wavelengths move faster, the long wavelengths or the short ones?

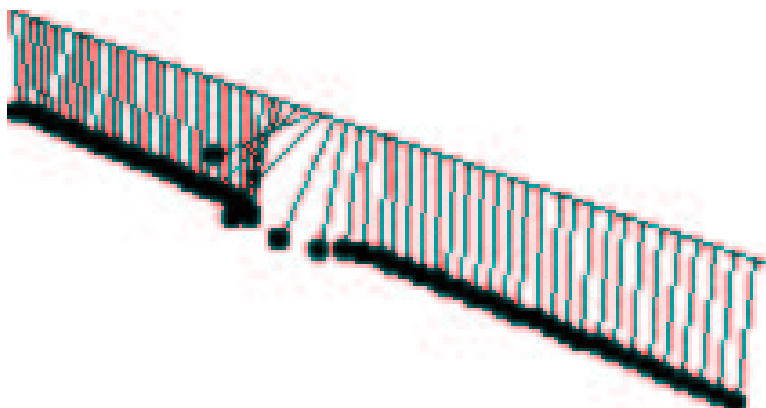
This is called *chirping*. Try making a sound that goes up in pitch at the end: what does it sound like?

- (c) Do these two answers agree with what you found for the dispersion relation in problem set 4?

Now change the number of string pieces (chunks) to 999 (the largest value allowed), and change the graph time skip back to 20.

- (d) Does the dispersion go away when you reduce the spacing δx in this way?

(5.6) Sine-Gordon Dispersion Relation.



An array of pendula connected by springs, in the continuum limit, obeys the Sine-Gordon equation

$$\partial^2 \phi / \partial t^2 = A \partial^2 \phi / \partial x^2 - B \sin(\phi).$$

with $\phi(x) = 0$ corresponds to the pendulum at position x along the array pointing downward. What is the dispersion relation $\omega(k)$ for small oscillations in this equation?

(A) $\omega(k) = (-B \pm \sqrt{B^2 - 4Ak^2}) / 2A$

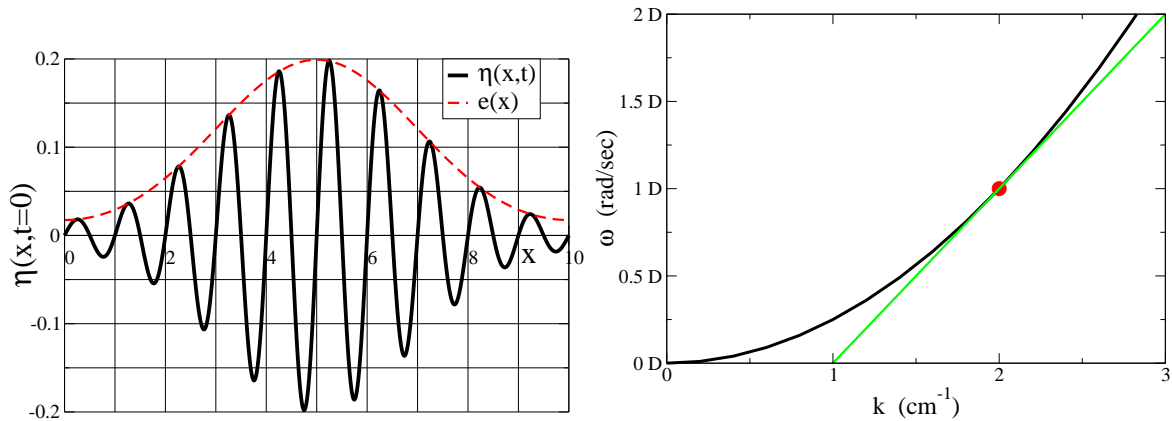
(B) $\omega(k) = \sqrt{Ak^2 - B \sin(\phi)}$

(C) $\omega(k) = \sqrt{Ak^2 - B}$

(D) $\omega(k) = \sqrt{Ak^2 + B}$

(E) $\omega(k) = \sqrt{Ak^2 + B \sin(\phi)}$

(5.7) Group and Phase Velocities.



A wave packet in an unusual optical fiber is traveling to the right. The wave packet has an envelope of shape $e(x)$ at time $t = 0$, and has a carrier wave with nodes at $x = 0.5\text{cm}$, $x = 1\text{cm}$, $x = 1.5\text{cm}$, ... as shown. The dispersion relation for the glass in this fiber is $\omega(k) = Dk^2$, as shown.* Which of the following formulas most closely describes $\eta(x, t)$, ignoring the slow spreading of the wave packet?

- (A) $\eta(x, t) \approx e(x - 4\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (B) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 1\pi Dt))$.
- (C) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (D) $\eta(x, t) \approx e(x - 2\pi Dt) \sin(2\pi(x - 4\pi Dt))$.
- (E) $\eta(x, t) \approx e(x - 1\pi Dt) \sin(2\pi(x - 2\pi Dt))$.
- (F) $\eta(x, t) \approx \exp(ikx - \omega t)$.

* It's not useful for this problem, but this is the dispersion relation for the wave equation $\partial^2\eta/\partial t^2 = -D\partial^4\eta/\partial x^4$.