

Physics 218: Waves and Thermodynamics

Fall 2003, James P. Sethna

Homework 7, due Monday Oct. 20

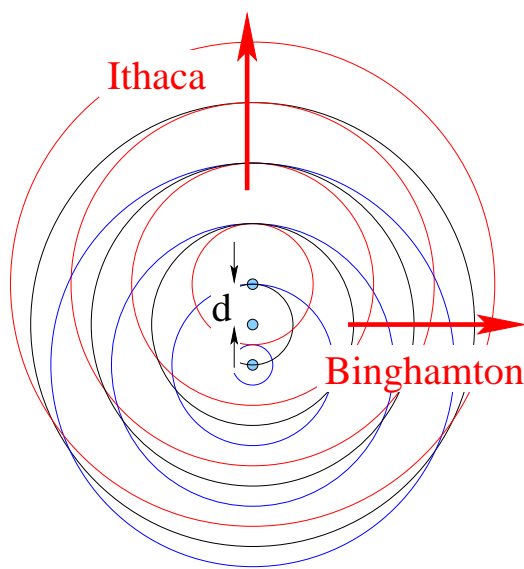
Latest revision: November 4, 2003, 11:31 am

Reading

Feynman, sections I.30 (Diffraction), II.30 (Crystals), and II.31 (Tensors)

Problems

(7.1) Antenna Array



A radio station south of Ithaca and west of Binghamton transmits with a carrier wave of wavelength λ . They use three antennas spaced equally along a north-south direction, separated by a distance $d = 2/3 \lambda$. The antennas are close together compared to the distance to either city. The radio station cleverly delays the signal in the antennas, so as to add a phase shift $\phi = 2\pi/3$ in the carrier wave between neighboring antennas. In particular, if we number the three antennas from south to north as $\{-1, 0, 1\}$, the radio signal is given by

$$A(r_0) (\exp[i(\omega t - kr_1 + \phi)] + \exp[i(\omega t - kr_0)] + \exp[i(\omega t - kr_{-1} - \phi)])$$

What is the ratio of the average intensity I_{av} of the radio signal in Ithaca, compared to I_0 , which one would find with only one antenna transmitting? In Binghamton?

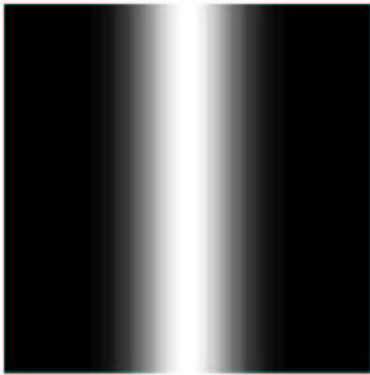
- (A) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 3$.
- (B) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 1$.
- (C) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 0$.
- (D) Ithaca: $I_{\text{av}}/I_0 = 3$; Binghamton $I_{\text{av}}/I_0 = 1$.
- (E) Ithaca: $I_{\text{av}}/I_0 = 3$; Binghamton $I_{\text{av}}/I_0 = 0$.

(7.2) Interference

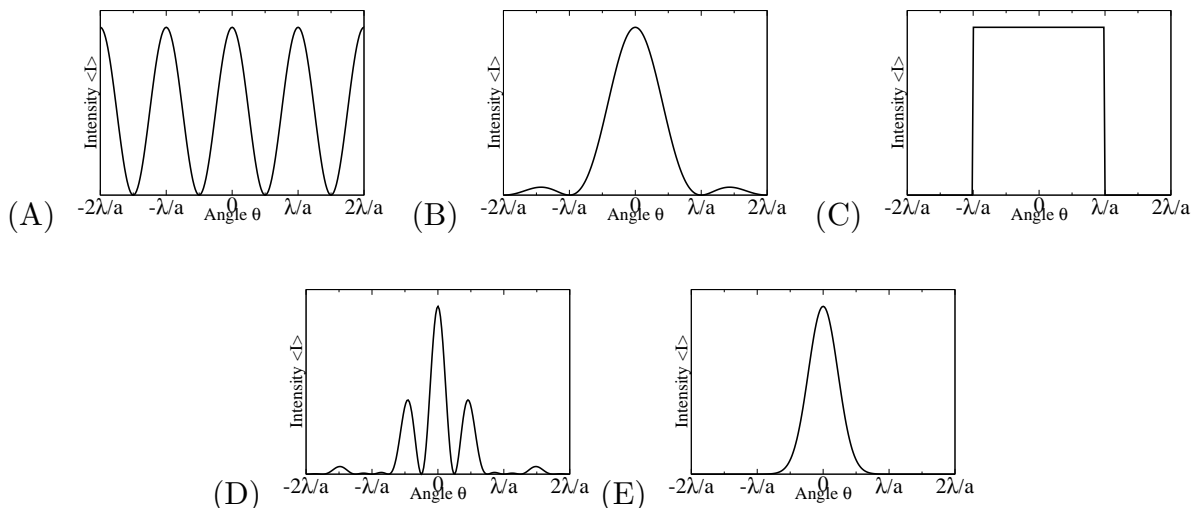
A coherent laser beam impinges on a slit of width a . An intensity pattern is viewed on a distant screen: the center has intensity I_0 and the peak width (distance between the nearest minima) is ΔY . The slit is broadened to $2a$. What is the new intensity $I_{doubled}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin \theta \approx \theta$.

- (A) $I' = 4I_0, \Delta Y' = \Delta Y/2.$
- (B) $I' = 2I_0, \Delta Y' = \Delta Y/2.$
- (C) $I' = 2I_0, \Delta Y' = \Delta Y/4.$
- (D) $I' = 4I_0, \Delta Y' = 2\Delta Y.$
- (E) $I' = 2I_0, \Delta Y' = 2\Delta Y.$

(7.3) Diffraction and Fourier Optics.



A glass slide is coated with soot, blocking all light except for a thin stripe across the center where the soot has been rubbed off. A laser beam of wavelength λ is aimed at the glass: the beam width is large compared to the slit width. A careful measurement of the light transmission immediately outside the glass shows that amplitude of the light has a Gaussian profile: it varies with x as $\exp(-2x^2/a^2)$, where a is the width (standard deviation) of the transparent strip and x is the distance from the center of the strip. Which of the following intensity patterns will be observed on a distant screen? (You may assume small angles.)



(7.4) Introduction to Tensors. In the next few weeks, we'll make heavy use of *tensors*. Tensors are a generalization of vectors and matrices: vectors v_i are one-index tensors, matrices M_{ij} are two-index tensors, and we'll be making use of three and four-index tensors like c_{ijkl} in our discussions of elasticity in solids. Just as for a vector or a matrix, a tensor c_{ijkl} is a multidimensional array of real numbers, one for each choice of i, j, k , and ℓ ranging from one to three.

(a) How many different real numbers are needed to specify a general four-index tensor?

In this problem, we introduce two particularly useful and important tensors. One is the Kronecker delta function δ_{ij} , which is one if $i = j$ and zero if $i \neq j$.

$$\delta_{ij} = 1 \quad \text{if } i = j \quad \delta_{ij} = 0 \quad \text{if } i \neq j \quad (7.4.1)$$

The other is the Levi-Civita symbol, or totally antisymmetric tensor, ϵ_{ijk} . It is defined by its value for $i = 1, j = 2, k = 3$,

$$\epsilon_{123} = 1 \quad (7.4.2)$$

and its antisymmetry property: it changes sign whenever two indices are permuted:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \quad (7.4.3)$$

It's easy to see that ϵ_{ijk} gives +1 if $\{ijk\}$ is an even permutation of $\{123\}$, -1 if it is an odd permutation, and zero if any two indices agree.

(b) Using equations (7.4.2) and (7.4.3), show that (specifically) $\epsilon_{223} = 0$; show also that $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$.

(c) Write out δ_{ij} as a matrix, with i labeling the row and j the column. What do you usually call this matrix? Write out ϵ_{ijk} as three matrices ϵ_{1jk} , ϵ_{2jk} , and ϵ_{3jk} , with i labeling the matrix, j the row and k the column. (We do not usually write out tensors in this way.)

One of the most common things we do to tensors is taking *outer products* and/or *contracting* them. The outer product of two tensors a_{ij} and b_{kl} , for example, is a tensor with four indices given by the product of the two: $d_{ijkl} = a_{ij}b_{kl}$. Contraction is done by setting two indices of a tensor (or an outer product of tensors) equal, and summing over all values of that repeated index: the new tensor has two fewer indices after contraction.

A familiar examples of a contraction is taking the trace of a matrix: $tr(M) = \sum_{i=1}^3 M_{ii}$. Three familiar examples of taking outer products and then contracting are the dot product of two vectors, $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^3 v_i w_i$, applying matrices to vectors $(M\mathbf{v})_i = \sum_{j=1}^3 M_{ij} v_j$, and multiplying matrices $(MN)_{ik} = \sum_{j=1}^3 M_{ij} N_{jk}$.

You notice that there are a lot of sums $\sum_{j=1}^3$ in the formulas above. In physics, we often make use of the **Einstein convention**, where summation (contraction) over repeated indices is implied. Hence if we write a_{ii} , the convention implies that we really meant the one-index tensor resulting from summing over i , $\sum_{i=1}^3 a_{ii}$.

- (d) Write the trace, dot product, matrix operating on a vector, and matrix multiplication examples above using the Einstein convention.
- (e) Give arguments for the following formulas involving δ_{ij} and ϵ_{ijk} . We use the Einstein convention.

$$\delta_{ii} = 3$$

(Easy.)

$$\epsilon_{ijk}\delta_{jk} = 0$$

(Consider two kinds of terms: $j = k$ and $j \neq k$.)

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

(Show that this is the sum of the squares of all of the elements of the tensor. How many non-zero elements are there?)

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$$

(Show that the left-hand-side is zero if $k \neq l$. Then compute it for $k = l = 3$, and argue from there.)

$$\epsilon_{ijm}\epsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

(Show the left-hand-side is zero except in the two cases ($i = k$ and $j = l$) and ($i = l$ and $j = k$). Then find the sign for the two cases.)

- (f) Write the cross product of two vectors \mathbf{v} and \mathbf{w} as the outer product contracted twice with the totally antisymmetric tensor.
- (g) (Optional: for the ambitious and inspired.) The determinant of a matrix M is anti-symmetric under interchange of any two rows or columns, and the determinant of the identity matrix is one. Argue (without doing messy calculations) that these are also true of the formula

$$\det M = (1/6) \epsilon_{ijk}\epsilon_{lmn} M_{il} M_{jm} M_{kn}.$$