Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 8, due Monday Oct. 27
Latest revision: October 21, 2003, 3:49 pm

Reading

Feynman, section I.30 (Diffraction), II.30 (Crystals), II.31 (Tensors), II.38-1/2 (Elasticity), II.39.1/5 (Strain and Elasticity)

Possibly also useful: Elmore & Heald, sections 3.1-3.3, 7.4

Prelim II

Prelim II is tentatively scheduled for Wednesday November 5, pending discussion about timing with the class. Prelim II will cover the higher-dimensional wave equations, interference and diffraction, tensors, elasticity theory, elastic waves, and electromagnetic waves. It will potentially include questions from the experimental lab *Microwaves and Optics*. It will be a similar format to the last exam.

Problems

(8.1) Diffraction Grating. A 10 cm wide diffraction grating with 10000 slits is used to measure the wavelengths emitted by hot hydrogen gas. (a) At what angles θ in the first order spectrum do we expect to find the two violet lines of wavelengths 434 and 410nm? (b) Same question but for second order.

(8.2) Thick Slits and Windows. Start up Huygens. Set the "Number of Slits" to one and the width a of the slit to 5 m. Notice the single-slit diffraction pattern on the right. Notice that the waves on the left look much like you'd expect for light coming in a window: light traveling along straight lines. How do we reconcile these two pictures?
Bring the screen in closer: try "X Screen" at 10 and 1 meter. (If you get too close, you'll begin to see my numerical method for generating the slit.) Now vary the wavelength. How much farther does the intensity look "window-like" at $\lambda = 0.35$ than for $\lambda = 0.7$? Finally, vary the slit width a. With $\lambda = 0.7$, what distance to the screen for $a = 2.5$ looks the same shape as 10m for $a = 5$?
Explain why you can hear around corners, but you can't see around corners.

(8.3) Double Thick Slit. Start up Huygens. Set d to 8m, and a to 2m. You should see a complicated interference pattern. Now set the number of slits to one. Is the single slit pattern the envelope of the double thick-slit pattern? Set the number of slits back to two, and set a to zero. Is the thin-slit pattern like the carrier wave?
Show that the intensity for a double-slit with distance d between the centers and width a for each slit is the product of the single-slit diffraction pattern of width a and the double thin-slit diffraction pattern. (Hint: Remember that the amplitude of the wave at the screen is proportional to the Fourier transform of the slit-opening function $E(\theta) = Re\left(Ae^{i\omega t}\tilde{f}(k\sin\theta)\right)$. You can find how the amplitude from the upper slit changes as it is translated upward by $d/2$ by using the properties of Fourier transforms under translation.)

(8.4) Interference A coherent laser beam impinges on a slit of width a. An intensity pattern is viewed on a distant screen: the center has intensity I_0 and the peak width (distance between the nearest minima) is ΔY. The slit is broadened to $2a$. What is the new intensity $I_{doubled}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin\theta \approx \theta$.

(A) $I' = 4I_0$, $\Delta Y' = \Delta Y/2$.

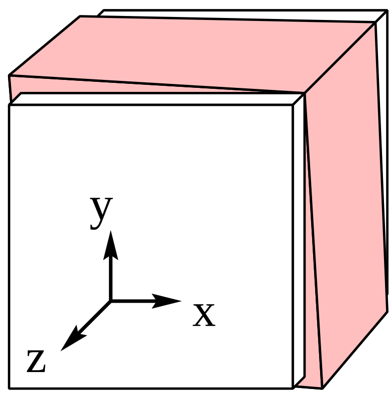
(B) $I' = 2I_0$, $\Delta Y' = \Delta Y/2$.

(C) $I' = 2I_0$, $\Delta Y' = \Delta Y/4$.

(D) $I' = 4I_0$, $\Delta Y' = 2\Delta Y$.

(E) $I' = 2I_0$, $\Delta Y' = 2\Delta Y$.

(8.5) Traction-free boundary condition.



An isotropic elastic medium is strained as shown above: it is compressed and stretched along different axes. The stress tensor is

$$\sigma_{ij} = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & -2a \end{pmatrix}.$$

The medium has a flat free surface perpendicular to the axis $\hat{\mathbf{n}}$. A free surface is a surface on which there is no traction, or forces, applied: for example, a surface exposed to vacuum (or approximately, to air). Knowing the stress tensor above, in which direction $\hat{\mathbf{n}}$ could the surface normal point?

(A) $\hat{\mathbf{n}} = (1, 0, 0)$

(B) $\hat{\mathbf{n}} = (0, 1, 0)$

(C) $\hat{\mathbf{n}} = (0, 0, 1)$

(D) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$

(E) $\hat{\mathbf{n}} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$

(F) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$

Draw $\hat{\mathbf{n}}$ and the strained cube, and convince yourself that the surface perpendicular to $\hat{n}$ would be free.

Related formulæ: $F_i/A = \sigma_{ij}\hat{\mathbf{n}}_j$, $F_i = \partial_j \sigma_{ij}$, $\sigma_{ij} = c_{ijkl}\epsilon_{kl} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$

(8.6) The Power of Tensors. Remember from last week the definitions of the two most important tensors: the Kronecker delta function $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and the totally antisymmetric tensor $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$ with $\epsilon_{123} = 1$. Remember the identities that you proved: $\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, and $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$. Notice that one can conveniently use tensor notation to write the gradient $(\nabla\psi)_i = \partial_i\psi$, divergence $\nabla \cdot \mathbf{a} = \partial_i a_i$, and curl $(\nabla \times \mathbf{a})_i = \epsilon_{ijk}\partial_j a_k$.

Use these formulas to prove the following vector identities (listed in the front of Jackson, *Classical Electrodynamics*):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla\psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi\mathbf{a}) = \nabla\psi \times \mathbf{a} + \psi\nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

HInt: You'll need to use the fact that the second derivative doesn't depend upon the order of variables ($\partial_i\partial_j f = \partial_j\partial_i f$) and the product rule ($\partial_i(f\,g) = (\partial_i f)g + f(\partial_i g)$).