Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 9, due Monday Nov. 3
Latest revision: November 4, 2003, 11:27 am

Reading

Feynman II.39.1/5 (Strain and Elasticity), I.51.3/4 (Waves in Solids and Surface Waves), II.39-3/4 (Elastic Motion), II.19 (The Principle of Least Action: not for credit), II.20 (Solutions of Maxwell's Equations in Free Space), and II.32 (Refractive Index of Dense Materials).

Possibly also useful: Elmore & Heald, sections 7.4, 7.5, 7.6, 8.1-8.5; don't worry about the obsolete notation of dyadics and stuff.

Prelim II

Prelim II is tentatively scheduled for Wednesday November 5, pending discussion about timing with the class. Prelim II will cover the higher-dimensional wave equations, interference and diffraction, tensors, elasticity theory, elastic waves, and electromagnetic waves. It will potentially include questions from the experimental lab *Microwaves and Optics*. It will be a similar format to the last exam.

Experimental Lab III

Interference and Diffraction, Monday evening 11/10 and Tuesday afternoon 11/11, Rock B26 and B30.

Problems

(9.1) Strain fields at large rotations. Show for a rotation about the z axis by an angle θ that the gradient of the displacement field $\partial_j u_i$ is

$$\vec{\nabla}\vec{u} = \begin{pmatrix} \cos(\theta)-1 & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta)-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Calculate the strain field $\epsilon_{ij}^{\text{approx}} = (1/2)(\partial_i u_j + \partial_j u_i)$ and show that it is not small. How large a rotation would give a 1% strain component to the strain tensor (and hence lead to plastic deformation)? Now calculate the true strain matrix including the "geometric nonlinearity" $\epsilon_{ij} = (1/2)(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k)$ and show that it is zero.

(9.2) Elastic Moduli.

In an isotropic material, only two elastic moduli are independent: all others can be written in terms of them. The tensor relation between strain and stress is most nicely written in terms of the two Lamé elastic constants λ and μ:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}. \tag{8.6.1}$$

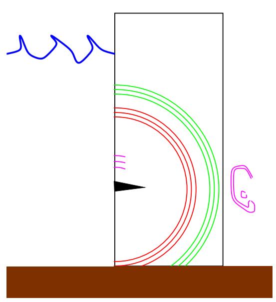
The constant μ is just the shear modulus (Feynman eqn. II.38.14); the constant $\lambda = B - 2\mu/3$, where B is the bulk modulus (Feynman's K, eqn. II.38.9).

(a) Show that bulk compression has three non-zero components: $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \Delta V/(3V)$. Knowing that the stress $\sigma_{xx} = B\Delta V/V$, check the formula for the bulk modulus given above.

If we put a force per unit area $\sigma_{xx} = F/A$ stretching our material in the x-direction, and we don't constrain the sides (so $\sigma_{ij} = 0$ except along x, $i = j = 1$), the material will stretch by an amount $\Delta L/L = (1/Y)F/A$, where Y is the Young's modulus. It also compresses in the y and z directions by Poisson's ratio* ν times the extension along the x direction, so the width and height change by $\Delta W/W = \Delta H/H = -\nu\Delta L/L$.

(b) Write the stress tensor σ_{ij} and the strain tensor ε_{kl} for this problem, as 3×3 matrices.

(c) Use your answers to part (b) and the equation (8.6.1) relating stress and strain for isotropic materials, solve for Young's modulus Y and Poisson's ratio ν in terms of λ and μ.



(9.3) Elastic Waves.

A small crack starts on the inside of a concrete dam, generating acoustic waves of all polarizations with wavelengths much shorter than the thickness D of the dam. An acoustical detector is positioned outside the dam directly opposite to the crack. The concrete can be assumed to be an isotropic medium with positive elastic constants λ and μ. What signal is expected in the acoustical detector?

(A) A transverse sound pulse, followed by a longitudinal sound pulse.
(B) A longitudinal sound pulse, followed by a transverse sound pulse.
(C) A transverse sound pulse only: sound is a transverse wave.
(D) A longitudinal sound pulse only: the transverse sound component will travel along the length and width of the dam, not across the thickness.
(E) A sound pulse after a time $t = D/\sqrt{Y/\rho}$, where Y is the Young's modulus of concrete.

Related formulæ: $\rho\partial^2 u_i/\partial t^2 = (\lambda + \mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$. With $\nabla\cdot\mathbf{u}_T = 0$, $c_T = \sqrt{\mu/\rho}$; with $\nabla\times\mathbf{u}_L = 0$, $c_L = \sqrt{(\lambda + 2\mu)/\rho}$

* We use ν for Poisson's ratio, as the engineers do, reserving σ for the stress tensor.

(9.4) Tensor Notation Review. Suppose $\mathbf{B} = \nabla\times\mathbf{A}$. Which of the following are correct formulas for $\mathbf{B}^2$? (For example, the energy contained in a magnetic field is $\mathbf{B}^2/8\pi$.)

(A) $\varepsilon_{ijk}\partial_j A_k \varepsilon_{i\ell m}\partial_\ell A_m$.
(B) $(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})(\partial_j A_k)(\partial_\ell A_m)$.
(C) $(\partial_j A_k)^2 - (\partial_j A_k \partial_k A_j)$.
(D) All of the above.
(E) None of the above.

(9.5) Elastic Traveling Wave. An isotropic elastic medium with density ρ and moduli λ and μ fills the half space $x > 0$. The boundary of this medium is wiggled with displacement field

$$\mathbf{u}(0, y, z) = (f(t), g(t), h(t)),$$

generating an elastic wave travelling to the right (positive x direction).
What is the displacement $\mathbf{u}(x, y, z, t)$ for $x > 0$?

(A) $\mathbf{u}(x, y, z, t) = (0, g(t - x/c), h(g - x/c))$.
(B) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{(\lambda + 2\mu)/\rho}), g(t - x/\sqrt{\mu/\rho}), h(t - x/\sqrt{\mu/\rho}))$.
(C) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - x/\sqrt{(\lambda + 2\mu)/\rho}), h(t - x/\sqrt{(\lambda + 2\mu)/\rho}))$.
(D) $\mathbf{u}(x, y, z, t) = (f(x - \sqrt{\mu/\rho}\, t), g(x - \sqrt{(\lambda + 2\mu)/\rho}\, t), h(x - \sqrt{(\lambda + 2\mu)/\rho}\, t))$.
(E) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - y/\sqrt{(\lambda + 2\mu)/\rho}), h(t - z/\sqrt{(\lambda + 2\mu)/\rho}))$.

(9.6) Waves on a Thin Wire. A plane wave of wave vector k passes along the $\hat{x}$ direction through a thin wire of radius W. The wire width W is thin compared to the wavelength, so $kW << 1$. The material making up the wire is isotropic, with elastic moduli λ and μ. The wave at $t = 0$ is approximately given by the real part of

$$\mathbf{u} = Ae^{i(kx-\omega t)}\left(1 - \nu\frac{k^2(y^2 + z^2)}{2}, -iky\nu, -ikz\nu\right)$$

where we use the engineering notation ν for Poisson's ratio $\nu = \lambda/2(\mu + \lambda)$.* This formula is correct up to terms of order k^3W^3. The wave is primarily longitudinal, for small k (the y and z components of $\mathbf{u}$ are smaller by a factor of kW than the x component). The wave is basically stretching and compressing the wire along the $\hat{x}$ direction, with a small correction.

(a) Ignoring for the moment the term proportional to k^2, show that the y and z components are just what one would expect from Poisson's ratio applied to the amount the wire is stretched along the x direction.

The k^2 term took me a long time to figure out the first year I taught this. I don't have a simple explanation for it, but without keeping it you get the wrong sound velocity even as $k \to 0$.

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* Feynman and E&H use σ for Poisson's ratio, which we use for the stress tensor.

(a) Compute the strain tensor $\varepsilon(x, y, z, t)$ for this displacement field, ignoring the geometric nonlinearity. Write it out as a 3×3 matrix.

(b) The wire is isotropic, with elastic moduli λ and μ. Write the stress tensor for the wire as a 3×3 matrix.

(c) (Not for credit: gluttons for punishment only.) Check that this displacement field satisfies Newton's law

$$\rho\partial^2 u_i/\partial t^2 = \partial_j \sigma_{ij}$$

and has zero stress at the surface of the wire *up to terms of order* k^3 *and* $k\omega^2$, with $\omega = ck$ and $c = \sqrt{Y/\rho}$.

Thus longitudinal sound down a thin wire travels with a speed of sound set by Young's modulus.