Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 10, due Monday Nov. 17
Latest revision: December 11, 2003, 4:34 pm

Reading

Feynman, I.1 Atoms in Motion, I.6 Probability, & I.43 Diffusion

Schroeder, chapter 1. Browse section 6.4 (Maxwell distribution of velocities).

Problems

Schroeder,

(1.12) *How Dilute is Air?* Assume small molecules are around 0.4 nm in diameter. Remember $PV = Nk_BT$ with $k_B = 1.381 \times 10^{-23}$, $P_{atm} \sim 1.013 \times 10^5$Pa, and $T \sim 300K$.

(1.18) *Molecular Velocities.* Remember the kinetic energy $(1/2)m\mathbf{v}^2$ of molecules equals $(3/2)k_BT$. The mass of a N_2 molecule is $m = 2m_{Ni} \sim 28m_p \sim (28gm/N_A) \times 1kg/1000gm = 4.65 \times 10^{-26}kg$.

(1.33) *P-V diagram.* Make a table with rows A, B, C, and "Whole Cycle" and columns "Work done on gas", "Change in Energy content of gas", and "Heat added to gas". Relevant formulas: Work on gas = -$\int P\,dV$; energy in gas $U = Nf(k_BT/2) = (Nf/2)PV$.

(1.60) *Frying Pan.* Do this three ways. (a) Guess the answer from your own experience. If you've always used aluminum pans, consult a friend.

(b) Use an argument analogous to Schroeder's equation (1.71) to get a rough answer. Note: For iron, the specific heat $c_p = 450J/kg \cdot C$, the density $\rho = 7900kg/m^3$, and the thermal conductivity $k_t = 80W/m{\cdot}C$. You need to transport heat $c_p\rho V\Delta T$ across an area $A = V/\Delta x$. How much heat will flow across that area per unit time, if the temperature gradient is roughly assumed to be $\Delta T/\Delta x$? How long δt will it take to transport the amount needed to heat up the whole handle?

(c) Roughly model the problem as the time needed for a pulse of heat at $x = 0$ on an infinite rod to spread out a distance equal to the length of the handle, and use the Greens function for the heat diffusion equation (problems 10.3 and 10.4 below). How long until the pulse spreads out a root-mean square distance $\sigma(t)$ equal to the length of the handle?

(10.1) Random walks in Grade Space. Let's make a simple model of the prelim grade distribution. Let's imagine a multiple-choice test of ten problems of ten points each. Each problem is identically difficult, and the mean is 70. How much of the point spread on the exam is just luck, and how much reflects the differences in skill and knowledge of the people taking the exam? To test this, let's imagine that all students are identical, and that each question is answered at random with a probability 0.7 of getting it right. What is the expected mean and standard deviation for the exam? (Work it out for one question, and then use our theorems for a random walk with ten steps.) A typical exam with a mean of 70 might have a standard deviation of about 15. What physical interpretation do you make of the ratio of the random standard deviation and the observed one?

(10.2) Probability Distributions. I'm assuming you're familiar with probabilities for discrete events (like coin flips and card games), but you probably haven't worked much with probability distributions for continuous variables (like human heights and atomic velocities). The three probability distributions most commonly encountered in physics are: (*i*) **Uniform:** $\rho_{\text{uniform}}(x) = 1$ for $0 \le x < 1$, $\rho(x) = 0$ otherwise; produced by random number generators on computers. (*ii*) **Exponential:** $\rho_{\text{exponential}}(t) = e^{-t/\tau}/\tau$ for $t \ge 0$, familiar from radioactive decay and used in the collision theory of gases. (*iii*) **Gaussian:** $\rho_{\text{gaussian}}(v) = e^{-v^2/2\sigma^2}/(\sqrt{2\pi}\sigma)$, describing the probability distribution of velocities in a gas, the distribution of positions at long times in random walks, the sums of random variables, and the solution to the diffusion equation.

(a) **Likelihoods.** What is the probability that a random number uniform on $[0, 1)$ will happen to lie between $x = 0.7$ and $x = 0.75$? That the waiting time for a radioactive decay of a nucleus will be more than twice the exponential decay time τ? That your score on an exam with Gaussian distribution of scores will be greater than 2σ above the mean? (Note: $\int_2^\infty (1/\sqrt{2\pi}) \exp(-v^2/2)\, dv = (1 - \text{erf}(\sqrt{2}))/2 \sim 0.023$.)

(b) **Normalization, Mean, and Standard Deviation.** Show that these probability distributions are normalized: $\int \rho(x) dx = 1$. What is the mean x_0 of each distribution? The standard deviation $\sqrt{\int (x - x_0)^2 \rho(x) dx}$? (Hint: $\int_{-\infty}^\infty (1/\sqrt{2\pi}) \exp(-x^2/2)\, dx = \int_{-\infty}^\infty x^2 (1/\sqrt{2\pi}) \exp(-x^2/2)\, dx = 1$).

(c) **Sums of variables.** Draw a graph of the probability distribution of the sum $x + y$ of two random variables drawn from a uniform distribution on $[0, 1)$. Argue in general that the sum $z = x + y$ of random variables with distributions $\rho_1(x)$ and $\rho_2(y)$ will have a distribution given by the *convolution* $\rho(z) = \int \rho_1(x) \rho_2(z - x)\, dx$.

(d) **Multidimensional probability distributions.** In statistical mechanics, we often discuss probability distributions for many variables at once (for example, all the components of all the velocities of all the atoms in a box). Let's consider just the probability distribution of one molecule's velocities. If v_x, v_y, and v_z of a molecule are all distributed with a Gaussian distribution with $\sigma = \sqrt{kT/M}$ (Feynman's equation 40.9, next week), then we describe the combined probability distribution as a function of three variables as the product of the three Gaussians:

$$\begin{aligned}\rho(v_x, v_y, v_z) =& 1/(2\pi(kT/M))^{3/2} \exp(-m\mathbf{v}^2/2kT) \\ =& \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_x^2}{2kT}}\right) \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_y^2}{2kT}}\right) \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_z^2}{2kT}}\right).\end{aligned}$$

Show, using your answer for the standard deviation of the Gaussian in part (b), that the mean kinetic energy is $kT/2$ per dimension. Show that the probability that the speed is $v = |\mathbf{v}|$ is given by a Maxwellian distribution

$$\rho_{\text{Maxwell}}(v) = \sqrt{2/\pi}(v^2/\sigma^3)\exp(-v^2/2\sigma^2).$$

(Hint: What is the probability that $|\mathbf{v}|$ is between v_r and $v_r + \Delta r$, for small Δr? The area of a sphere of radius R is $4\pi R^2$.)

(e) Assuming the probability distribution for the z component of velocity given in part (d), $\rho(v_z) = \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_z^2}{2kT}}\right)$, give the probability density that an N_2 molecule will have a vertical component of the velocity equal to the escape velocity from the Earth (about 10 km/sec, if I remember right). Do we need to worry about losing our atmosphere? (Hint: this is closely related to Schroeder's problem 1.18.) Optional: Try the same calculation for H_2, where you'll find a substantial leakage. You'll want to know that there are 3×10^{16} seconds in a billion years, and molecules collide (and scramble their velocities) many times per second. That's why Jupiter has hydrogen gas in its atmosphere, and Earth does not.

(10.3) Thermal Diffusion. The rate of energy flow in a material with thermal conductivity k_t and a temperature field $T(x, y, z, t) = T(\mathbf{r}, t)$ is $\mathbf{J} = -k_t \nabla T$ (see Feynman eq. 43.41). Energy is locally conserved, so the energy density E satisfies $\partial E/\partial t = -\nabla \cdot \mathbf{J}$.

(a) If the material has constant specific heat c_p and density ρ, so $E = c_p \rho T$, show that the temperature T satisfies the diffusion equation $\partial T/\partial t = \frac{k_t}{c_p \rho}\nabla^2 T$. (See Schroeder, problem 1.62).

(b) By putting our material in a cavity with microwave standing waves, we heat it with a periodic modulation $T = \sin(kx)$ at $t = 0$, at which time the microwaves are turned off. Show that amplitude of the temperature modulation decays exponentially in time. How does the amplitude decay rate depend on wavelength $\lambda = 2\pi/k$?

(10.4) Heat Diffusion Spot. The diffusion equation for the heat density in a two-dimensional sheet is

$$\partial q/\partial t = K(\partial^2 q/\partial x^2 + \partial^2 q/\partial y^2).$$

(a) **Diffusion in Two Dimensions.** Show that if $f(x, t)$ satisfies the diffusion equation in one dimension, then $f(x, t)f(y, t)$ solves the diffusion equation in two dimensions. (Related formulæ: Product Rule, $\partial fg/\partial z = \partial f/\partial z\, g + f\, \partial g/\partial z$.)

(b) **The heat spot.** A screen of thermal diffusion constant K is heated at $x = y = 0$ and $t = 0$ by a thin laser beam pulse. The total heat deposited is Q. Use part (A) and the Greens function for the one dimensional diffusion equation to derive the equation for $q(x, y, t)$, the heat density after a time t. What is the root-mean-square radius $r(t) = \sqrt{\langle x^2 + y^2 \rangle}$ for this spot? (Related formulæ: $\partial \rho/\partial t = D\partial^2 \rho/\partial x^2$; If $\rho(x, 0) = \delta(x)$, $\rho(x, t) = G(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$ and $\langle x^2 \rangle = 2Dt$; $\langle f(\mathbf{z}) \rangle = \int f(\mathbf{z})\rho(\mathbf{z})\, d^D z$.)