Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 11, due Monday Nov. 24
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Reading

Feynman, I.39 The Kinetic Theory of Gases, I.40 Principles of Statistical Mechanics, I.41 The Brownian Movement, & I.42 Applications of Kinetic Theory.

Schroeder, "Very Large Numbers" subsection of section 2.4, 2.5 (Ideal Gas) & 3.1 (Temperature).

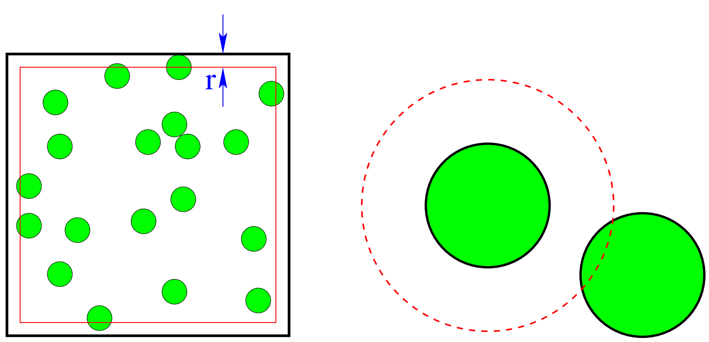
Web reading

Introduction to the Cosmic Microwave Background Radiation:
http://background.uchicago.edu/∼whu/beginners/introduction.html
http://background.uchicago.edu/∼whu/intermediate/intermediate.html,
especially the parts Acoustic Oscillations, Angular Peaks, and First Peak.

Problems

Schroeder,
(3.2) *Zeroth law.*
(3.3) *Entropy graphs.*

(11.1) Entropy and Hard Spheres.



We can improve on the realism of the ideal gas by giving the atoms a small radius. If we make the potential energy infinite inside this radius ("hard spheres"), the potential energy is simple (zero unless the spheres overlap, which is forbidden). Let's do this in two dimensions.

A two dimensional $L \times L$ box contains an ideal gas of N hard disks of radius $r \ll L$ (left figure). The disks are dilute: the summed area $N\pi r^2 \ll L^2$. Since the disks cannot be within r of the edges of the box, let A be the effective volume allowed for the first disk in the box: $A = (L - 2r)^2$.

(a) **Configuration Space Volume for Hard Disks.** The area allowed for the second disk is $A-\pi(2r)^2$ (right figure), ignoring the small correction when the excluded region around the first disk overlaps the excluded region near the walls of the box. The area allowed for the n^{th} disk is $A-(n-1)\pi(2r)^2$, ignoring corrections for the overlaps of the excluded regions. Let configuration space $\mathbf{X}$ be the $2N$ dimensional space of positions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots\mathbf{x}^{(N)}$. Write an expression for the volume $\Omega_{\mathbf{X}}$ of allowed zero-energy configurations of hard disks, in the configuration space $\mathbf{X}$, ignoring the overlapping excluded regions. (*Related formulæ*: For a 3D ideal gas, $\Omega_{\mathbf{P}} = (\pi' s)(2mE)^{(3N-1)/2}$, $\Omega_{\mathbf{X}} = V^N/N!$.)

(b) **Statistical Mechanical Entropy for Hard Disks.** It's now easy to write the configurational entropy, $S_{\mathbf{X}}$ for the hard disks of part (a) as a sum over n. Use the "Math truth" below to find a formula for the entropy that does not involve a sum over n, accurate to first order in the area of the disks πr^2. (*Related formulæ*: $S = k_B \ln(\Omega)$; **Simpson's Rule:** $n! \approx (n/e)^n\sqrt{2\pi n}$; **Math Truth**: To first order in ϵ, $\sum_{n=1}^{N} \log\left(A-(n-1)\epsilon\right) = N\log\left(A-(N-1)\epsilon/2\right)$.)

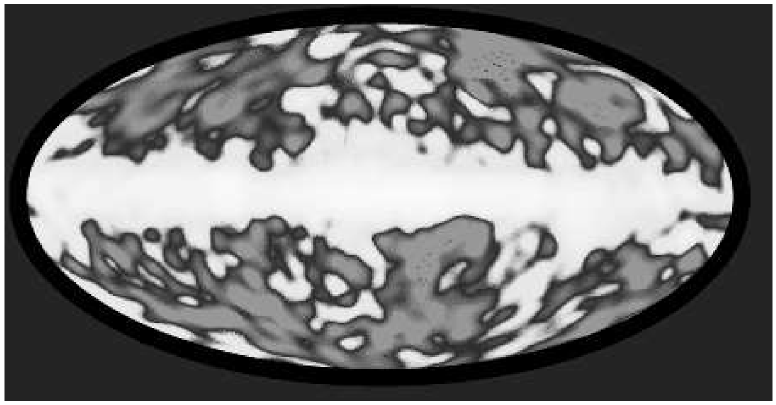
(c) **Pressure for Hard Disks.** Assume the hard-disk configurational entropy is $S_{\mathbf{X}} = Nk_B\log(A-Nb)$ for some area b, representing the effective excluded area due to the other disks. (Your answer to (b) won't quite have this form, but it's a good approximation, up to an overall N-dependent constant.) Just as for the ideal gas, the internal energy U is purely kinetic, and the kinetic energy and momentum-space entropy depend only on temperature and not on volume. So, if we isothermally expand this hard-disk gas from initial area A_1 to A_2, the internal energy doesn't change: $\Delta U = Q + W = 0$, so the heat Q added to the gas equals $-W$, the work done by the gas expanding against the external pressure P. By differentiating with respect to A_2, find the pressure for the hard-sphere gas. (Hint: for $b = 0$ it should reduce to the ideal gas law.) (*Related formulæ*: $W = -\int_{A_1}^{A_2} P dA$ and $\Delta S = Q/T$ (Thermodynamic Entropy). For a 3D ideal gas, $PV = Nk_BT$ and $U = 3/2\,Nk_BT$.)

(11.2) Waves and the Birth of the Universe. (Large thanks to Ira Wasserman: errors are of course my own.)
Our universe started very hot and dense, in what we call the Big Bang. This auspicious starting point is what sets our arrow of time.

Because the universe is expanding, the light emitted back then has redshifted (due to the Doppler effect), so the immensely hot and bright origin of the universe now resides in a microwave background radiation that you'd get from a black body at a temperature of 3 K. We've learned a lot about our universe recently by carefully measuring the differences between the temperatures of this radiation as we look in different directions in the night sky.

Figure (11.2.1) shows these tiny fluctuations in temperature.* The fluctuations in temperature represent noisy thermal waves in the early universe.

* Actually, it shows these fluctuations after a dipole term has been subtracted out.



(11.2.1) Microwave background radiation map. Variation in temperature of the microwave background radiation, after the constant term and the dipole term are subtracted out, from COBE, the Cosmic Microwave Background Explorer. The fluctuations are about one part in 100,000. The bright stripe at the equator is our galaxy.

Because it was still very hot, all the hydrogen in the universe was still ionized. Light doesn't travel very far in ionized gases (it accellerates the charges and scatters from them): the light and matter remained in equilibrium with one another until the universe was around 300,000 years old, when it got cold enough for the electrons and protons to combine into hydrogen.

Before 300,000 years, the combined light-and-matter density satisfied a wave equation:

$$(1+R)\partial^2\Theta/\partial t^2 = (c^2/3)\nabla^2\Theta, \tag{1}$$

where c is the speed of light in vacuum, Θ is the temperature fluctuation $\Delta T/T$, t is "conformal" time (treat it as regular time), and R is the contribution of matter to the density. Θ can be viewed also as the energy density fluctuations $\Delta e/e$ where $e = U/V$ is the energy density: denser regions are hotter. After recombination, the light was able to travel directly (albeit red-shifted) to our cameras. So, the microwave background radiation is giving us a snapshot of the temperature fluctuations of the universe at age 300,000 years.

(a) What is the speed of sound in this gas?

Let's derive equation (1).

(b) The dominant contribution to the pressure of this combined light-and-matter mixture is due to the light pressure. Feynman (section 39.3) shows that the photon gases satisfy $PV^{4/3} = C$, where C is some constant. (Photons are quantized light particles: you'll learn about them more in Modern Physics.) The bulk modulus B is defined by

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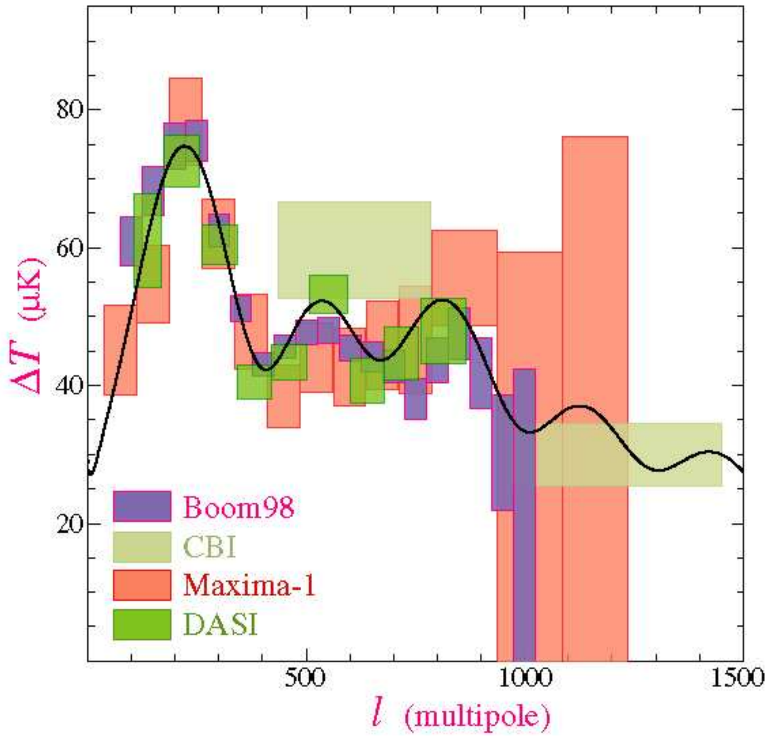
The dipole comes from the Doppler effect of *our* motion. Einstein's theory states that all motion is relative: the laws of physics don't depend upon how fast the Sun is moving with respect to the distant galaxies. But that doesn't mean that the distant galaxies (or, even better, the glow from the Big Bang) doesn't have a particular velocity! We can measure our velocity with respect to the universe by using this dipole.



(11.2.2) Wave vector dependence of microwave radiation pattern. Variation in temperature of the microwave background radiation, decomposed into spherical harmonics. Spherical harmonics are like a Fourier transform, but for angles: you can think of ℓ for the multipole as roughly corresponding to wavenumber k of the corresponding temperature fluctuation in the universe when it became transparent to photons (at recombination) (From Wayne Hu's Web site, above).

$\Delta P = -B\Delta V/V$. Show that the bulk modulus is $4P/3$. Feynman also shows that $PV = U/3$. Let P_0 be the average light pressure, U_0 be the average energy in the light in an initial volume V_0. Show (trivially) that $B = (4/9)U_0/V_0$ where U_0/V_0 is the average photon energy density.

(c) The total mass density for the wave equation ρ in the early universe has three important contributions. First, there is the regular mass of particles (mostly baryons) M_{baryon}/V_0. Then there is the energy density of the photons divided by c^2 (remember $E = mc^2$?), $U_0/(V_0c^2)$. Finally, there is a contribution due to the pressure P_0/c^2 (this is really a component of a stress-energy tensor...) Show that the total density is $\rho = M_{\text{baryon}}/V_0 + 4U_0/(3V_0c^2)$.

(d) Derive equation (1) above from $\rho\partial^2 P/\partial t^2 = B\nabla^2 P$. What is the formula for R?

A theory called "inflation" predicts that at very early times the universe was left in a state which we can think of as being uniform in temperature and density, but with a random velocity field. Let's derive what the density field $\Theta(\mathbf{x})$ should look like at time $t = 300,000$ years.

(e) Consider first an initial standing-wave perturbation $\Theta(\mathbf{x}, t) = \tilde{\Theta}_k \sin(\mathbf{k}\cdot\mathbf{x})\sin(\omega_k t)$. (Of course the universe started with a superposition of many such standing waves with different $\mathbf{k}$.) What is ω_k? The energy density fluctuation is zero at $t = 0$, as inflation predicts. At what times will this wave have maximum fluctuations? Which values of $k = |\mathbf{k}|$ will have largest fluctuations at $t = 300,000$ years? Show that odd multiples of the first peak are maxima, while even multiples are minima. Assuming for simplicity that $R = 0$ (photon-dominated mass density), give the wavelength of the first peak, in light years.

Our picture of the background radiation (first above) is a cross section of the original radiation at a sphere given by the 10 billion years since recombination (modulo corrections due to the age of the universe). Since the data is on a sphere, they need to decompose our data into spherical harmonics: the constant ℓ in the wave-vector figure (II.2.2) roughly corresponds to wave number k.

(f) Is twice the ℓ value of the first maximum in figure (11.2.2) a maximum or a minimum? Does that agree with your conclusion for part (e)? What about three times the first maximum? (The full theory includes other effects which shift the peak positions.)