Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 12, due Wednesday Dec. 3
Latest revision: December 11, 2003, 4:36 pm

Reading

Feynman, I.44 Laws of Thermodynamics, I.45 Illustrations of Thermodynamics, & I.46 Ratchet and Pawl

Schroeder, 2.6 (Entropy), 3.1 (Temperature), & 4.1 (Heat Engines. Browse the rest of chapter 4 (Engines and Refrigerators).

For further reading (much more advanced)

Freeman J. Dyson, "Time without end: Physics and biology in an open universe", *Reviews of Modern Physics* **51**, 447 (1979). (Download from prola.aps.org).

Problems

Schroeder,

(3.10) *Entropy and Ice Cubes.* The latent heat of ice is 80 cal/g, and the specific heat of water is $c_p = 1cal/(gm \cdot K)$; one calorie is 4.186 J.

(3.16) *Entropy and bits.*

(12.1) Life and the Heat Death of the Universe.

Freeman Dyson discusses how living things might evolve to cope with the cooling and dimming we expect during the heat death of the universe.

Dyson models an intelligent being as a heat engine that generates a fixed entropy ΔS per thought. (This correspondence of information with entropy is a standard idea from computer science.)

(a) **Energy needed per thought.** Assume that the being draws heat Q from a hot reservoir at T_1 and radiates it away to a cold reservoir at T_2. What is the minimum energy Q needed per thought, in terms of ΔS and T_2? You may take T_1 very large. (*Related formulæ*: For Carnot engine, $\Delta S = Q_2/T_2 - Q_1/T_1 = 0$; First Law: $Q_1 - Q_2 = W$ (energy is conserved).)

(b) **Time needed per thought to radiate energy.** Dyson shows, using theory not important here, that the power radiated by our intelligent–being–as–heat–engine is no larger than CT_2^3, a constant times the cube of the cold temperature.* Write an expression for the maximum rate of thoughts per unit time dH/dt (the inverse of the time Δt per thought), in terms of ΔS, C, and T_2.

* The constant scales with the number of electrons in the being, so we can think of our answer Δt as the time per thought per mole of electrons.

(c) **Number of thoughts for an ecologically efficient being.** Our universe is expanding: the radius R grows roughly linearly in time t. The microwave background radiation has a characteristic temperature $\Theta(t) \sim R^{-1}$ which is getting lower as the universe expands: this red-shift is due to the Doppler effect. An ecologically efficient being would naturally try to use as little heat as possible, and so wants to choose T_2 as small as possible. It cannot radiate heat at a temperature below $T_2 = \Theta(t) = A/t$. How many thoughts H can an ecologically efficient being have between now and time infinity, in terms of ΔS, C, A, and the current time t_0?

(d) **Time without end: Greedy beings.** Dyson would like his beings to be able to think an infinite number of thoughts before the universe ends, but consume a finite amount of energy. He proposes that his beings need to be profligate in order to get their thoughts in before the world ends: he proposes that they radiate at a temperature $T_2(t) \sim t^{-3/8}$ which falls with time, but not as fast as $\Theta(t) \sim t^{-1}$. Show that with Dyson's cooling schedule, the total number of thoughts H is infinite, but the total energy consumed U is finite.

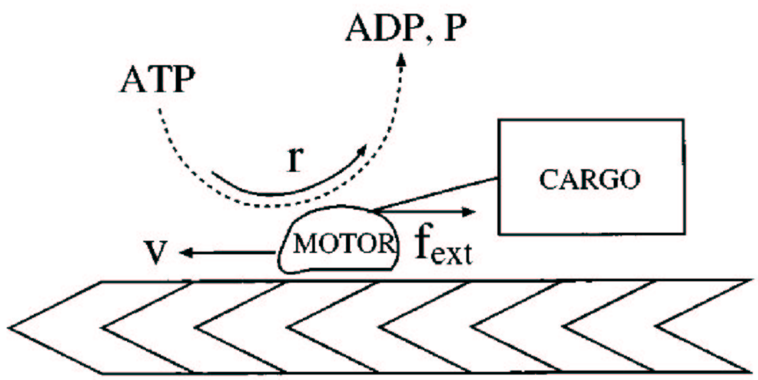

Figure (12.2.1) Cartoon of a motor protein, from Jülicher, Ajdari, and Prost, *Rev. Mod. Phys.* **69**, 1269 (1997). As it carries some cargo along the way (or builds an RNA or protein, ...) it moves against an external force f_{ext} and consumes r ATP molecules, which are hydrolized to ADP and phosphate (P).

(12.2) Ratchet and Molecular Motors.

Feynman's ratchet and pawl discussion obviously isn't so relevant to machines you can make in your basement shop. The thermal fluctuations which turn the wheel to lift the flea are too small to be noticable on human length and time scales (you need to look in a microscope to see Brownian motion). On the other hand, his discussion turns out to be surprisingly close to how real cells move things around. Physics professor Michelle Wang studies these molecular motors in the basement of Clark Hall.

Inside your cells, there are several different molecular motors, which move and pull and copy (figure 12.2.1). There are molecular motors which contract your muscles, there are motors which copy your DNA into RNA and copy your RNA into protein, there are motors which transport biomolecules around in the cell. All of these motors share some common

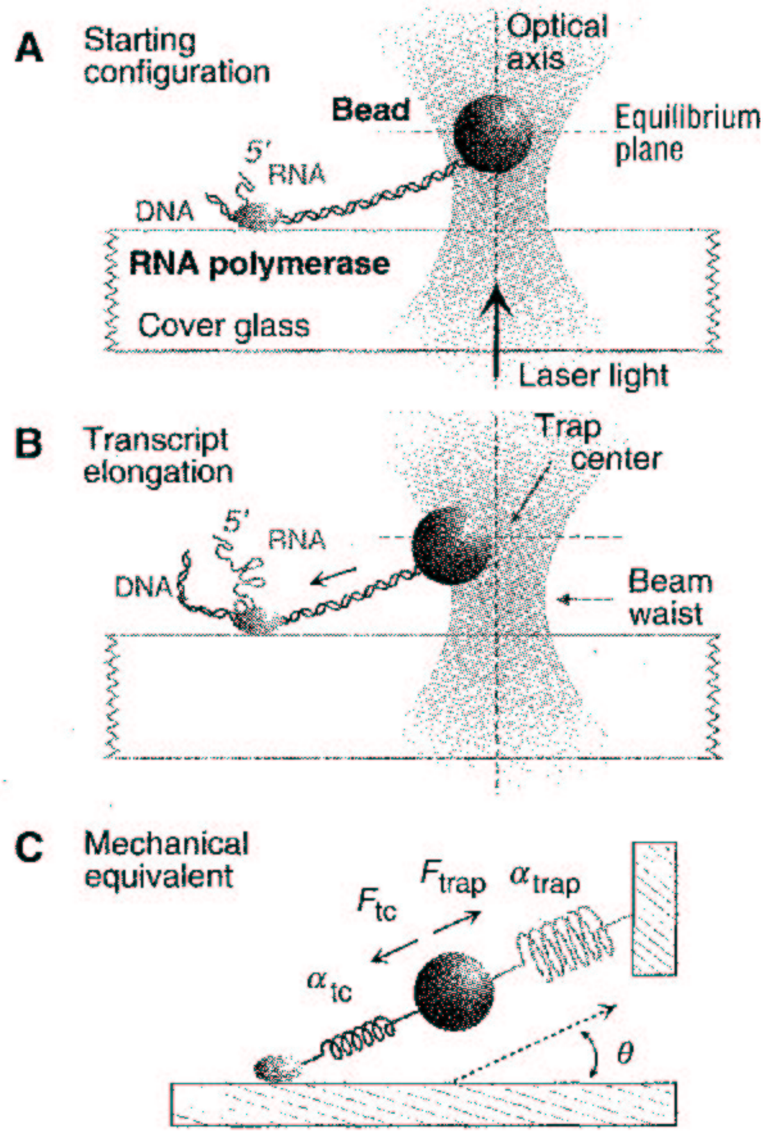

Figure (12.2.2) Cartoon of Professor Wang's early laser tweezer experiment, (Yin, Wang, Svoboda, Landick, Block, and Gelles, *Science* **270**, 1653 (1995)). (A) The laser beam is focused at a point (the "laser trap"); the polystyrene bead is pulled (from dielectric effects) into the intense part of the light beam. The "track" is a DNA molecule attached to the bead, the motor is an RNA polymerase molecule, the "cargo" is the glass cover slip to which the motor is attached. (B) As the motor (RNA polymerase) copies DNA onto RNA, it pulls the DNA "track" toward itself, dragging the bead out of the trap, generating a force resisting the motion. (C) A mechanical equivalent, showing the laser trap as a spring and the DNA (which can stretch) as a second spring.

features: (1) they move along some linear track (microtubule, DNA, ...), hopping forward in discrete jumps between low-energy positions, (2) they consume energy (burning ATP or NTP) as they move, generating an effective force pushing them forward, and (3) their mechanical properties can be studied by seeing how their motion changes as the external

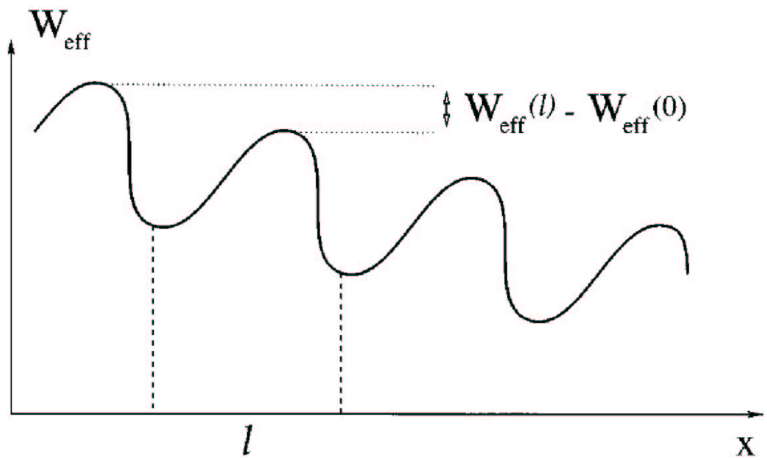

Figure (12.2.3) The effective potential for moving along the DNA (from Prost, above). Ignoring the tilt W_e, Feynman's energy barrier ϵ is the difference between the bottom of the wells and the top of the barriers. The experiment changes the tilt by adding an external force pulling ℓ to the left. In the absence of the external force, W_e is the (Gibbs free) energy released when one NTP is burned and one RNA nucleotide is attached.

force on them is changed (figure 12.2.2).

For transcription of DNA into RNA, the motor moves on average one base pair (A, T, G or C) per step: $\Delta\ell$ is about 0.34nm. We can think of the triangular grooves in the ratchet as being the low-energy states of the motor when it is resting between steps. The barrier between steps has an asymmetric shape (figure 12.2.3), just like the energy stored in the pawl is ramped going up and steep going down. Professor Wang showed (in a later paper) that the motor stalls at an external force of about 27 pN (pico-Newton).

(a) At that force, what is the energy difference between neighboring wells due to the external force from the bead? (This corresponds to $L\theta$ in Feynman's ratchet.) Let's assume that this force is what's needed to balance the natural force downhill that the motor develops to propel the transcription process. What does this imply about the ratio of the forward rate to the backward rate, in the absence of the external force from the laser tweezers, at a temperature of $300K$, (from Feynman's discussion preceding equation 46.1)? ($k_B = 1.381 \times 10^{-23}$J/K).

The natural force downhill is coming from the chemical reactions which accompany the motor moving one base pair: the motor burns up an NTP molecule into a PP_i molecule, and attaches a nucleotide onto the RNA. The net energy from this reaction depends on details, but varies between about 2 and 5 times 10^{-20} Joule. This is actually a Gibbs free energy difference, but for this problem treat it as just an energy difference.

(b) The motor isn't perfectly efficient: not all the chemical energy is available as motor force. From your answer to part (a), give the efficiency of the motor as the ratio of force-times-distance produced to energy consumed, for the range of consumed energies given.

(12.3) Carnot Refrigerator. Our refrigerator is about $2m \times 1m \times 1m$, and has insulation about $3cm$ thick. The insulation is probably polyurethane, which has a thermal conductivity of about 0.02 W/(m K). Assume that the refrigerator interior is at 270K, and the room is at 300K.

(a) How many watts of energy leak from our refrigerator through this insulation?

Our refrigerator runs at 120 V, and draws a maximum of 4.75 amps. The compressor motor turns on every once in a while for a few minutes.

(b) Suppose (*i*) we don't open the refrigerator door, (*ii*) the thermal losses are dominated by the leakage through the foam and not through the seals around the doors, and (*iii*) the refrigerator runs as a perfectly efficient Carnot cycle. How much power on average will our refrigerator need to operate? What fraction of the time will the motor run?

(12.4) Entropy of Glasses. Glasses aren't really in equilibrium. In particular, they do not obey the third law that the entropy S goes to zero as the temperature approaches absolute zero. Experimentalists measure a "residual entropy" by subtracting the entropy change from the known entropy $S_{\text{equilibrium}}(T)$ at high temperatures (say, in the ordinary equilibrium liquid state):

$$S_{\text{residual}} = S_{\text{equilibrium}}(T) - \int_0^T \frac{dQ}{T\,dT} dT.$$

Usually, one calls dQ/dT the specific heat C of the material, but we're being fussy:

(a) If you put a glass in an insulated box, it will warm up (very slowly) because of microscopic atomic rearrangements which lower the potential energy. So, glasses don't have a well-defined temperature or specific heat. In particular, the heat flow upon cooling and on heating $\frac{dQ}{dT}(T)$ won't precisely match (although their integrals will agree by conservation of energy). By using the second law (entropy can only increase), show that the residual entropy measured on cooling is always less than the residual entropy measured on heating.*

(b) The residual entropy of a glass is about k_B per molecular unit. It's a measure of how many different glassy configurations of atoms the material can freeze into (section I.46-4). In a molecular dynamics simulation with one hundred indistinguishable atoms, and assuming that the residual entropy is $k_B \log 2$ per atom, what is the probability that two coolings to zero energy will arrive at equivalent atomic configurations (up to permutations)? In a system with 10^{23} molecular units, with residual entropy $k_B \log 2$ per unit, about how many coolings would be needed to arrive at the same configuration twice?

* See Steve Langer's paper, *Phys. Rev. Lett.* **61**, 570 (1988), although M. Goldstein noticed it earlier.