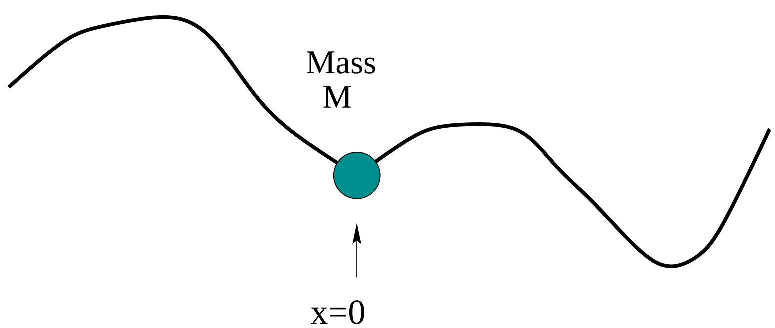
Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 1, due Monday Sept. 1
Latest revision: December 18, 2003, 3:15 pm

Reading

Elmore & Heald, section 1.1
Feynman, I.22-5/6

Problems

(1.1) Bead on a String.



A bead of mass M is attached at $x = 0$ to a string of mass density λ_0 stretched with horizontal tension τ. The string extending to the left (negative x) has height $\eta_1(x,t)$, and to the right it has height $\eta_2(x,t)$. The bead has height $y(t) = \eta_1(0,t) = \eta_2(0,t)$. What is the formula for the acceleration of the bead? (Draw the free body diagram!)

(A) $d^2y/dt^2 = (\tau/M)\partial^2\eta/\partial x^2$

(B) $d^2y/dt^2 = (\tau/M)(\eta_2(0,t) - \eta_1(0,t))$

(C) $d^2y/dt^2 = (\tau/M)(\eta_1(0,t) - \eta_2(0,t))$

(D) $d^2y/dt^2 = (\tau/M)\left(\frac{\partial\eta_2}{\partial x}|_{x=0} - \frac{\partial\eta_1}{\partial x}|_{x=0}\right)$

(E) $d^2y/dt^2 = (\lambda_0/2\tau)\left(\frac{\partial^2\eta_1}{\partial x^2}|_{x=0^-} + \frac{\partial^2\eta_2}{\partial x^2}|_{x=0^+}\right)$

(1.2) Fourier Series. The laws for the motion of stretched strings, of the surface of water, of sound, and of electromagnetic radiation are called *wave equations* because they all have special solutions of the form of sinusoidal waves. That is, a string with initial height $A\sin(kx)$ or $B\cos(kx)$ will time evolve in a particularly simple way. We need to review some mathematics about sinusoidal waves.

(a) (Review) What is the wavelength of the shape $A\sin(kx)$, where x is the distance measured along the string?

We call k the *wave vector* for the wave.

(b) (Review) Suppose we study a stretched string with the ends at $x = 0$ and $x = L$ held fixed at height $y = 0$. Calculate the values k_m at which $\eta(x) = A\sin(kx)$ satisfies these two boundary conditions. (To be specific, let $m - 1$ be the number of zeros, or nodes, for $y(x)$ inside the string, not including the boundaries. For this problem, all values of k_m should be positive.)

In this course, we will make extensive use of complex numbers. In quantum mechanics, the waves really involve complex amplitudes, but for this course the complex numbers are just a way to make the mathematics simpler: our waves will be the real parts of complex waves. You should remember the formula

$$\exp(ikx) = \cos(kx) + i\sin(kx). \tag{1.2.1}$$

Thus cosine waves are the real part of the complex wave $\exp(ikx)$.

(c) (Review) If k is positive, for what smallest positive value of x_0 is the real part of $\exp(ik(x - x_0))$ a sine wave, $\sin(kx)$?

The Fourier series for a function $y(x)$ is an expansion in terms of sinusoidal waves. Elmore and Heald concentrate on the Fourier sine and cosine expansions. In our work, we'll use complex Fourier series. Suppose we have a function $y(x)$ defined on $0 \le x \le L$, with $y(0) = y(L)$. (This is called *periodic boundary conditions*, since we can make y into a periodic function by placing new copies side-by-side over each period L.) Various mathematical theorems tell us that we can write $y(x)$ as an infinite series

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x). \tag{1.2.2}$$

in terms of the complex sinusoidal waves $\exp(ik_m x)$ which satisfy the same boundary condition.

(d) Show that $y(x) = \exp(ik_m x)$ satisfies $y(0) = y(L)$ if $k_m = 2\pi m/L$ (here k_m may be positive or negative). Are these the same wave vectors as you found in part (b)?

The formula for the complex Fourier series coefficients $\tilde{y}_m$ of a function $y(x)$ in an interval of length L is

$$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx. \tag{1.2.3}$$

Mathematical theorems tell us that the sum in equation (1.2.2) converges to $y(x)$ if we use the coefficients from equation (1.2.3). Also, the coefficients are unique: if the coefficients aren't all the same, the functions are different.

(e) Use equation (1.2.3) to compute the Fourier coefficients $\tilde{y}_m$ with $m = -1$, 0, and 1, for $\sin(2\pi x/5)$, in an interval of length $L = 5$. Check this using the well-known formula $\sin(\theta) = (\exp(i\theta) - \exp(-i\theta))/2i$. Without using the formula (1.2.3), but using the fact that the coefficients are unique, give *all* the Fourier coefficients for $7\cos(18\pi x/5)$, again with $L = 5$. (Hint: what's the well-known formula for $\cos(\theta)$?)

Decomposing a function into a Fourier series, equation (1.2.2), is like writing a vector as a sum $\mathbf{v} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$. Instead of a three-dimensional space of vectors, we have an infinite-dimensional space of functions. Our "unit vectors" are the complex exponential waves $\exp(ik_m x)$. Finding the coefficients, equation (1.2.3), is like taking the dot product to find the coefficient in the expansion, $a_x = \mathbf{v} \cdot \hat{\mathbf{x}}$, etc., except that the dot product of two complex functions is generalized to an integral of one times the complex conjugate of the other,

$$f \cdot g = (1/L) \int_0^L f(x) g^*(x)\, dx. \tag{1.2.4}$$

The dot products of different unit vectors $\hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$: they are orthogonal to one another. Also, the unit vectors are normalized, so $\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1$.

(f) Show that our Fourier series functions $\exp(ik_m x)$ with $k_m = 2\pi m/L$ are normalized under the dot product (1.2.4). Show that any two different Fourier series functions $m \neq n$ are orthogonal under the dot product (1.2.4).

(1.3) Fourier Series: Computer Lab.

Download the executables for *Fourier* from the bottom of the course home page. (The Windows version works, as far as I know. The Linux version may not: let me know if you have success or not, and which version of Linux you run.) When you start it up, you'll find at left a graph of the function $y(x)$, and at right the Fourier series $\tilde{y}(k_m)$ for the function. Thus to get m in $\tilde{y}_m$, take the coordinate along the horizontal axis and multiply by $L/2\pi$.

Use *Fourier* to check your answers for problem 1.2(e). In particular, (i) set $L = 5$, (ii) find m and x_0 to plot $y(x) = \sin(2\pi x/5)$, and (iii) read off the Fourier coefficients from the right graph. (You can zoom in with the mouse. Black is the real part, red the imaginary part.) Do they agree with those you found in 1.2(e)? Then (iv) find m, x_0, and the amplitude A to generate $7\cos(18\pi x/5)$, (v) double the number of points N repeatedly until $y(x)$ looks like a smooth sine wave, and (vi) read off the Fourier transform. Does it agree?

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 2, due Monday Sept. 8
Latest revision: December 18, 2003, 3:15 pm

Reading

Elmore & Heald, sections 1.2, 1.3, 1.4, 1.5, 1.6, 1.7
Feynman, sections I.22-5, I.22-6, I.23 Feynman, sections I.50-1/4

Problems

Elmore & Heald, page 7, problem 1.2.3 (stationary initial condition), and page 13, problem 1.3.2 (a).

Quick ones.

Elvis. Elvis notices that his A string on his guitar is off pitch: it is vibrating at 445 Hz. He wants it to sound at 440 Hz.
(a) Is his guitar string sharp (too high pitch) or flat (too low)?
(b) Elvis twists the little knob at the top of the string to tune it to 440 Hz. Did he tighten or loosen the tension?
(c) By what percentage does he change the tension?

Sympathetic Vibration. Consider two strings of equal mass density and length. When the strings are near each other, starting string 1 vibrating in its fundamental mode causes string 2 to vibrate in its third (n=3) natural mode. What is the ratio of the tension of string 1 to string 2?

Numerical Derivatives. The angle $\theta(t)$ of a pendulum is measured at three different times: $\theta(1.8) = 0.72$, $\theta(2.0) = 0.74$, and $\theta(2.2) = 0.82$. Estimate the accelleration $\partial^2\theta/\partial t^2$ at $t = 2.0$.

Big ones.

(2.1) Solving the Wave Equation Numerically

Consider a string of length L that is shaken up and down at the left end $\eta(0, t) = f(t)$ and is fixed in position $\eta(L, t) \equiv 0$ at the right end.

$$\frac{\partial^2\eta}{\partial t^2} = c^2\frac{\partial^2\eta}{\partial x^2} \tag{1}$$

To solve this equation numerically, we must discretize the string into chunks of size δx in space, and take small, discrete time steps δt in time.

(a) Derive the approximate formula for the second derivative

$$\frac{\partial^2 \eta}{\partial x^2} \approx \frac{\eta(x+\delta x, t) - 2\eta(x,t) + \eta(x-\delta x, t)}{\delta x^2} \tag{2}$$

from the approximate formula for the first derivative

$$\frac{\partial \eta}{\partial x}(x_0) \approx \frac{\eta(x_0+\epsilon/2) - \eta(x_0-\epsilon/2)}{\epsilon}. \tag{3}$$

(Hint: pick $\epsilon = \delta x$ and $x_0 = x \pm \delta x/2$. It may help to draw a picture of where you are evaluating the first and second derivatives.)

(b) Applying this approximate formula to the wave equation (1), show that we can write the future position of the string in terms of the past and present. If our wire is broken up into N chunks of size $\delta x = L/N$,

$$x_0 \equiv 0, \quad x_1 = \delta x, \quad \ldots \quad x_N = N\delta x \equiv L \tag{4}$$

show that

$$\eta(x_i, t+\delta t) \approx 2\eta(x_i,t) - \eta(x_i, t-\delta t) + (c\,\delta t/\delta x)^2 \left(\eta(x_{i+1},t) - 2\eta(x_i,t) + \eta(x_{i-1},t)\right). \tag{5}$$

Notice that this equation applies for $i = 1, \ldots N-1$, but not for $i = 0$ or $i = N$. These *boundary conditions* have to be supplied separately: in our case, fixed on the right, forced on the left.

(c) Write a program (using Matlab, Mathematica, a spreadsheet, or any other method of your choice) to solve this wave equation with $L = 15m$, $c = 2m/s$, $\delta x = 0.5m$, $\delta t = 0.1s$, and

$$f(t) = \exp(-(6-t)^2/4). \tag{6}$$

Use the evolution equation (5) and the initial conditions

$$\eta(x_i, 0) \equiv \eta(x_i, -\delta t) \equiv 0. \tag{6}$$

When should the pulse center hit the right end of the string? Plot the pulse shape when the center is partway to the wall, when your analysis says it should be hitting the wall (notice the numerical error in our calculation), and after it is reflected. Where do you think the energy is stored when the pulse is at the wall?

(2.2) Fourier Series and Gibbs Phenomenon

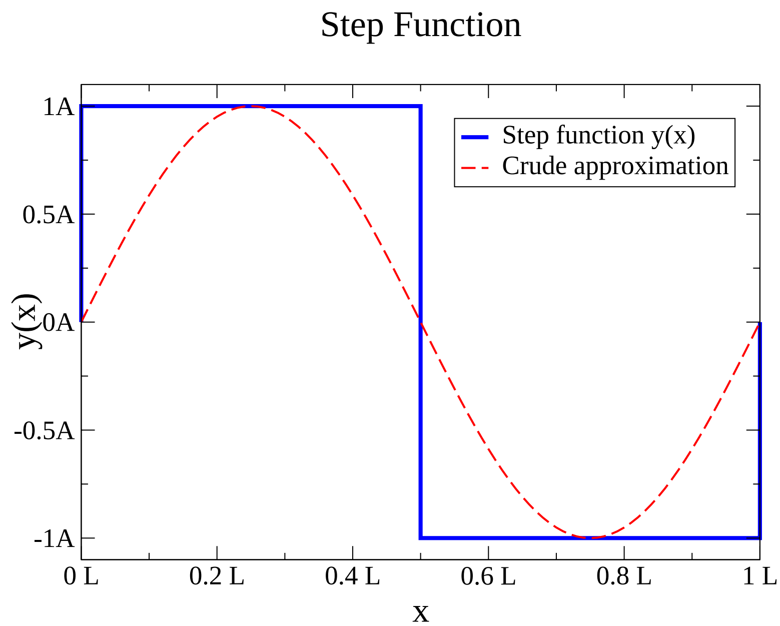


Figure 2.2.1 Step Function.

We defined complex Fourier series in the last problem set:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x), \tag{1.2.2}$$

$$\tilde{y}_m = (1/L) \int_0^L y(x) \exp(-ik_m x) dx, \tag{1.2.3}$$

with $k_m = 2\pi m/L$. In this problem set, we'll look at the Fourier series for a couple of simple functions, the step function (above) and the triangle function.

Consider a function $y(x)$ which is A in the range $0 < x < L/2$ and minus A in the range $L/2 < x < L$ (shown above). It's a kind of step function, since it takes a step downward at $L/2$.*

(a) As a crude approximation, the step function looks a bit like a chunky version of a sine wave, $A \sin(2\pi x/L)$. In this crude approximation, what would the complex Fourier series be?

(b) Show that the odd coefficients for the complex Fourier series of the step function are $\tilde{y}_m = -2Ai/(m\pi)$ (m odd). What are the even ones? Check that the coefficients $\tilde{y}_m$ with $m = \pm 1$ are close to those you guessed in part (a).

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* It can be written in terms of the standard Heaviside step function $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$, as $y(x) = A\,(1 - 2\Theta(x - L/2))$.

(c) Setting $A = 2$ and $L = 10$, plot the partial sum of the series equation (1.2.2) for $m = -n, -n+1, \ldots, n$ with $n = 1$, 3, and 5. (You'll likely need to combine the coefficients $\tilde{y}_m$ and $\tilde{y}_{-m}$ into sines or cosines, unless your plotting package knows about complex exponentials.) Does it converge to the step function? If it is not too inconvenient, plot the partial sum up to $n = 100$, and concentrate especially on the overshoot near the jumps in the function at 0, $L/2$, and L. This overshoot is called the Gibbs phenomenon, and occurs when you try to approximate functions $y(x)$ with discontinuities.

One of the great features of the Fourier series is that it makes taking derivatives and integrals easy.

(d) Show that the Fourier series of the derivative of a function $y'(x) = dy/dx$ is $\tilde{y'}_m = ik_m \tilde{y}_m$. Show, for $m \neq 0$, that the Fourier series for the integral of a function $y(x)$ is $\tilde{y}_m/(ik_m)$.

What does the integral of our step function look like? Let's sum the Fourier series for it!

(e) Consider the Fourier series whose coefficients are $\tilde{y}_m/(ik_m)$, where $\tilde{y}_m$ is the complex Fourier series you defined in part (b), and where you can set the $m = 0$ coefficient to zero. This series should sum to an integral of the step function. Do partial sums up to $\pm m = n$, with $n = 1$, 3, and 5, again with $A = 2$ and $L = 10$. Would the derivative of this function look like the step function? If it's convenient, do $n = 100$, and notice there are no overshoots.

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 3, due Monday Sept. 15
Latest revision: December 18, 2003, 3:15 pm

Reading

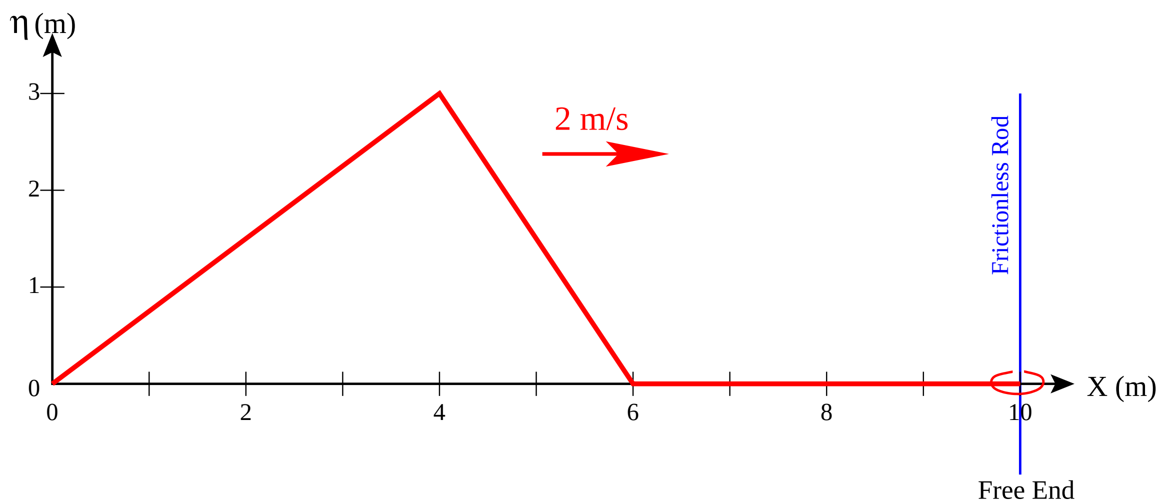
Elmore & Heald, sections 1.6, 1.7, 1.8, 1.9

Feynman, I.23, I.49-1/2, I.50-1/4

Problems

Elmore & Heald, page 38, problems 1.8.1 (Steel wire), 1.8.4 (Continuity equation for energy density).

(3.1) Traveling Wave on a String. The figure below shows a traveling wave propagating to the right on a string at time $t = 0$. The tension is $8N$ and the string has mass per unit length $2kg/m$. The string has length $10m$ and has a fixed end at $x = 0$ and a free end at $x = 10m$.



(a) Draw a graph of the transverse velocity (chunk velocity) of the wave at time $t = 0$, labeling your axes and giving units.
(b) Draw graphs of the energy density, the power, and the momentum density of the wave at $t = 0$.
(c) Draw graphs of the height of the wave and its transverse velocity at $t = 4$ seconds. Show that the total energy is the same as that at $t = 0$. Is the total momentum the same?
(d) Draw a graph of the transverse velocity at $x = 5$ as a function of time, from $t = -1$ second to $t = 4$ seconds.
(e) A new pulse of the same shape but twice as high and half as wide is sent down the wire. The energy density plot will be half as wide (why?) and how many times as tall? How much will the total energy change?

(3.2) Fourier wave. A musical instrument playing a note of frequency ω_1 generates a pressure wave $P(t)$ periodic with period $2\pi/\omega_1$: $P(t) = P(t+2\pi/\omega_1)$. The complex Fourier series of this wave is zero except for $n = \pm 1$ and ± 2, corresponding to the fundamental ω_1 and the first overtone. At $n = 1$, the Fourier amplitude is $2 - i$, at $n = -1$ it is $2 + i$, and at $n = \pm 2$ it is 3. What is the pressure $P(t)$?

(A) $\exp((2+i)\omega_1 t) + 2\exp(3\omega_1 t)$

(B) $\exp((2\omega_1 t))\exp(i(\omega_1 t)) * 2\exp(3\omega_1 t)$

(C) $\cos 2\omega_1 t - \sin\omega_1 t + 2\cos 3\omega_1 t$

(D) $4\cos\omega_1 t - 2\sin\omega_1 t + 6\cos 2\omega_1 t$

(E) $4\cos\omega_1 t + 2\sin\omega_1 t + 6\cos 2\omega_1 t$

(3.3) Pythag: Resonance.

We'll be using a few computer simulations to illustrate ideas from the course. We don't expect long writeups. Download the program pythag, from the course Web site (or directly from links at the bottom of

http://www.physics.cornell.edu/sethna/teaching/sss/pythag/pythag.htm).

The download will contain several programs: look for pythag.exe.

Play with the program for a while. Observe the effects of fixed, free, and reflectionless boundary conditions. Using fixed boundary conditions on both sides, and "Wave" forcing on the left, hit "Initialize" and "Run": the system is periodically forced on the left boundary at a frequency Ω and with an amplitude A that you can set on the Configure menu. Change Ω to 10 rad/s, A to 0.01, and the time to run on the main controls to 10 s. (You need to hit Enter to get changes to register: the number turns red to warn you.) Notice that the string wiggles under the external forcing, but the amplitude never gets very large.

Now, using the tension τ, the mass per unit length μ_1 (what Elmore & Heald calls λ_0), and a length L (all given under the Configure menu), find the frequencies ω_m of the standing waves. Change the frequency of the forcing frequency Ω to the frequency ω_1 of the fundamental mode, and reduce A to 0.002. How does the amplitude in the fundamental mode build up? The small graph on the lower left shows the height Y of the center of the string (our η) as a function of time: it should be oscillating with an increasing amplitude $\eta_{max} \sim t^\zeta$ as the resonance builds up. Do the peaks seem to be growing linearly in time ($\zeta = 1$), or quadratically ($\zeta = 2$), or what?

In your writeup, we'd like to see the frequency that you forced the program to excite the fundamental, and a brief, qualitative description of the growth of the oscillation peaks in time.

(3.4) Pythag: Energy and Power.

Let a pulse be traveling down the string at the velocity of sound $\eta(x, y) = f(x - vt)$. Use the fact that this is a traveling wave to derive a formula giving the ratio of the potential energy density to the kinetic energy density. Restart pythag (or select DEFAULT on the presets), and verify your formula. (For the writeup, just note the maximum amplitude for the kinetic and potential energy densities KE and PE.)

Derive a formula relating the power and the total energy density u for a traveling wave. Verify your formula with pythag.

Be sure to remember: these two formulas only apply to traveling waves!

(3.5) Fourier Transforms

In problem set 1, we defined the complex Fourier series of a function confined to an interval $(0, L)$. Waves on strings, rods, and in boxes and tanks are all confined to defined regions, but many waves are unconfined. Fourier transforms are like Fourier series, except that the range of the function goes from $(-\infty, \infty)$. The Fourier transform of a function $y(x)$ is another function $\tilde{y}(k)$:

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x) \exp(-ikx)\, dx \qquad (3.5.1)$$

and you can retrieve the original function back by using the inverse Fourier transform:

$$y(x) = (1/2\pi) \int_{-\infty}^{\infty} \tilde{y}(k) \exp(ikx)\, dk \qquad (3.5.2).$$

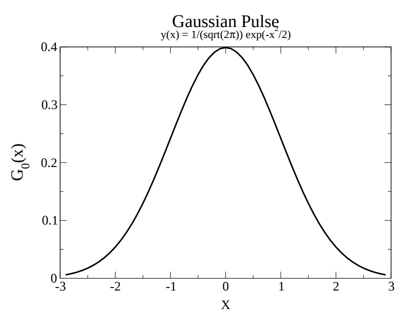


Figure 3.5.1 Gaussian Pulse centered at $x_0 = 0$ of width $\sigma = 1$.

The famous function

$$G(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-(x - x_0)^2/2\sigma^2) \qquad 3.5.3$$

is usually called a normal distribution or a normalized Gaussian. It peaks at x_0, and as $x - x_0 \to \pm\infty$ the Gaussian dies rapidly to zero (because of the exponential of minus x^2). In fact, it starts getting small at about $|x - x_0| = \pm\sigma$. Thus the function is a pulse of width σ centered at x_0. It is of fundamental important in probability theory, in quantum mechanics, and in statistical mechanics (last month of this course). It is also a good example of a pulse (like the sound you might get from slapping your hand on the table). Let's call $G_0(x)$ the Gaussian with mean x_0=0 and width $\sigma = 1$, pictured above.

(a) Show that the Fourier transform $\tilde{G}(k) = \exp(-ikx_0)\tilde{G}_0(\sigma k)$, by changing variables in equation (3.5.1) from x to $z = (x - x_0)/\sigma$. Notice that you should not need to do any integrals!

The Gaussian $G(x)$ has some nice properties: the integral (norm) $\int_{-\infty}^{\infty} G(x)\,dx = 1$, the mean $\int_{-\infty}^{\infty} xG(x)\,dx = x_0$, the variance (or square of the width) $\int_{-\infty}^{\infty} (x-x_0)^2 G(x)\,dx = \sigma^2$. Also, the Fourier transform of the standard Gaussian $G_0(x)$ of width one and mean zero $\tilde{G}_0(k) = \exp(-k^2/2)$. The derivation for three of these four formulas is a bit tricky, so treat them as given.

(b) Using the formulas above and your answer for part (a), give the general formula for the real and imaginary parts of $\tilde{G}(k)$. Draw pictures of the answer for $\sigma = 2$ and $x_0 = 4$, going from $k = -2.5$ to $k = 2.5$.

The Fourier transform of a Gaussian centered at zero is another Gaussian! It's not normalized, though: its height is always one at $k = 0$.

(c) In general, the value of the Fourier transform $\tilde{y}(k)$ at $k = 0$ gives which basic property of y, the norm, mean, or variance?

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 4, due Monday Sept. 22
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Experimental Lab I

Standing Waves, Monday evening 9/15 and Tuesday afternoon 9/16, Rock B26 and B30.

Computer Lab

Fourier Lab, Monday evening 9/22 and Tuesday afternoon 9/23, Rock B3 (hidden around the corner in the basement).

Reading

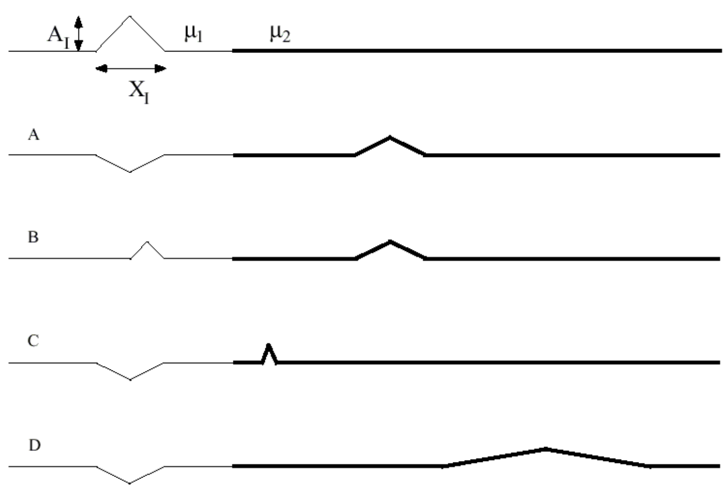
Elmore & Heald, sections 4.1, 4.7, 12.1, 12.2

Feynman I.47, I.48-1/4, I.50-1/4

Problems

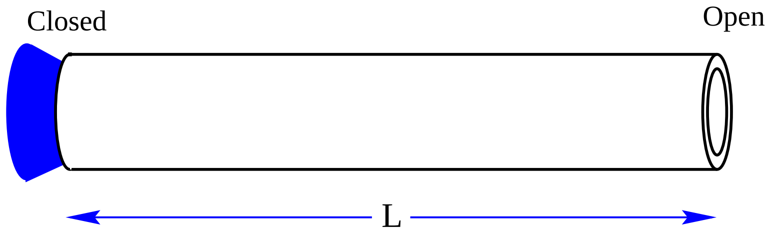
Elmore & Heald, page 41, problems 1.9.2 (Bead on a String). Use Pythag to see that the pulse indeed does not stay the same shape: (a) set reflectionless boundary conditions on both sides, (b) force with a pulse on the left, (c) make $\mu_2 = 20$, (d) make $X_{12} = 4.99$ and $X_{23} = 5.01$, (e) set the graph time step to one and the amplitude A to 0.1.

(4.1) Reflection and Transmission.



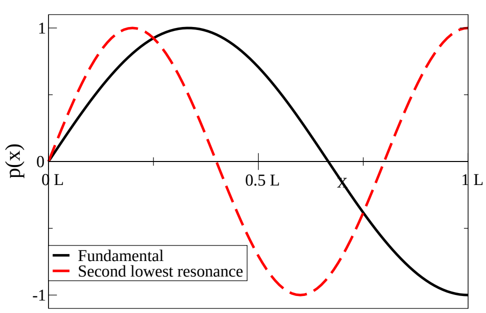
A pulse of height A_I, width X_I, travels on a string of mass density μ_1 and is incident on a string of mass density $\mu_2 = 9\mu_1$. The strings are joined together, and have the same tension. Which picture correctly describes the string after the pulse has interacted with the junction between the two strings? The pictures are drawn to scale.

(4.2) Air in a tube.

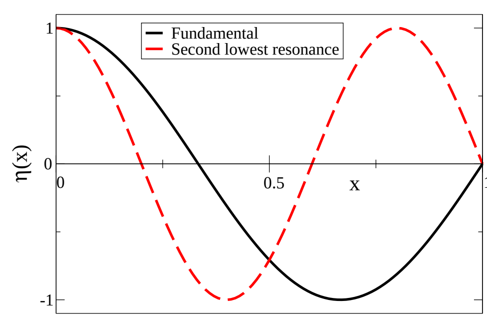


A tube of air of length L is closed on the left-hand side and open on the right. Which pictures represent the *pressure* change $p(x)$ from atmospheric pressure for the lowest two resonant frequencies in this tube? (The solid line is the fundamental, the dashed line represents the second lowest tone.)

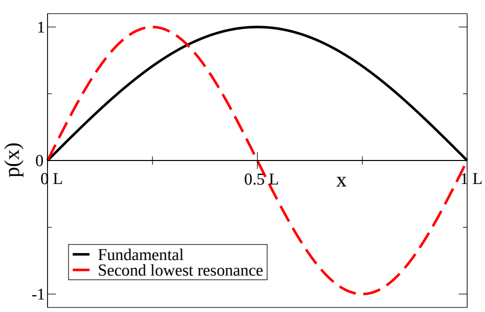
(A)



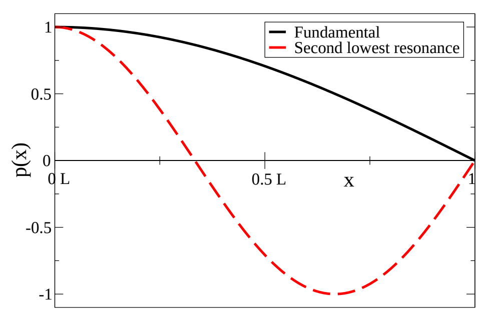
(B)



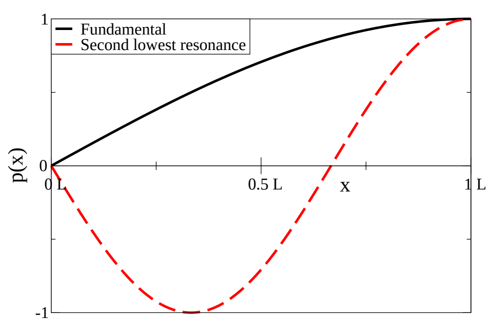
(C)



(D)



(E)



(4.3) Pythag: Reflection and Transmission.

Start up pythag, and choose the PRESET for STEPDOWN. The string comes in two pieces, whose mass densities μ_1 and μ_2 can be read off the Configure menu. The thickness of the lines roughly corresponds to the mass densities. To repeat a run, first Initialize, then Run.

Notice some qualitative facts. (1) The pulses leaves the simulation without reflection at the boundaries. I had to carefully match impedences at the boundary to avoid reflections. (2) Notice that the string is continuous and has a continuous derivative at the junction. (You can slow the pulse by lowering "graph time skip" on the Configure menu.) (3) How do the widths of the reflected and transmitted pulses compare to the incident pulse? How about their duration in time, passing by a particular place? Why should their durations agree? (4) Which is largest, the incident, reflected, or transmitted pulse? According to your transmission formula, should that always be the case if $\mu_2 < \mu_1$? (5) Does the reflected pulse invert or not? How about the reflected pulse for STEPUP? By taking the limits where the mass density ratio goes to zero and infinity, argue why this is related to reflection at fixed and free boundary conditions.

(4.4) Atoms: Dispersion and the 1-D Crystal.

In lecture we derived the equation of motion for the longitudinal displacements u_n of the nth atom in a chain of atoms connected by springs,

$$\frac{\partial^2 u_n}{\partial t^2} = (K/M)[u_{n+1} - 2u_n + u_{n-1}] \tag{4.4.1}$$

where K is the spring constant and M is the atomic mass. Assume a plane-wave solution

$$u_n = \sin(kna - \omega t) \tag{4.4.2}$$

where a is the equilibrium distance between atoms.

(a) **Dispersion Relation.** Plug in the trial solution equation (4.4.2) into equation (4.4.1). Rewrite $u_{n\pm1}$ by expanding the sines, $\sin(k(n \pm 1)a - \omega t) = \sin((kna - \omega t) \pm ka) = \sin(kna - \omega t)\cos(ka) \pm \cos(kna - \omega t)\sin(ka)$ and hence write your equation in the form $-\omega^2 (BLAH) = f(k)(BLAH)$. Solve for the *dispersion relation*, the frequency $\omega(k) = \sqrt{-f(k)}$ for each wave-vector k in our one-dimensional crystal.

(b) **Continuum Limit.** What is the speed of sound for our chain at long wavelengths? To be specific, what is $\omega(k)/k$ (the phase velocity) as the wavelength goes to infinity and hence $k \to 0$? (A Taylor series under the square root might be useful.)

In the regular wave equation, where $\omega(k) = c\,k$, both the group velocity $d\omega/dk$ and the phase velocity $\omega(k)/k$ give the speed of sound, independent of k.

(c) Using $K/m = a = 1$, plot the dispersion relation $\omega(k)$ for $-\pi/a < k < \pi/a$. On a second graph, plot the group velocity and the phase velocity for $0 < k < \pi/a$. Which velocity is larger, for this dispersion relation?

(4.5) Decibels. The power difference between sound A and B in dB is $10\log_{10}(P_A/P_B)$, where P is the power (energy per unit area). (Bels are named after Alexander Graham Bell, the telephone guy; a decibel is a tenth of a Bel.) The threshold of hearing is around zero decibels (0 dB). The threshold of pain is about 120 dB, and corresponds to a power of about $1W/m^2$. From this and your knowledge of air and sound, estimate the amplitude of the vibration of your eardrum at the threshold of audibility. (The bulk modulus of air B is about $1.4 \times 10^5\,N/m^2$; the density of air is about $1.2\,kg/m^3$; the speed of sound in air is about $340\,m/s$; a typical sound frequency might be $1000\,Hz$.) Compare this with other natural scales of length: which is it closest to, the size of your ear, the width of a hair in your cochlea, the width of a cell, the width of an atom, ...

(4.6) Fourier Series, Fourier Transforms, and FFTs.

In problem set 1, we introduced the Fourier series for periodic functions of period L,

$$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx, \tag{1.2.3}$$

where $k_m = 2\pi m/L$. The Fourier series, we saw explicitly in problem set 2, can be resummed to retrieve the original function:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x). \tag{1.2.2}$$

In problem set 3, we introduced the Fourier transform for functions on the infinite interval

$$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x)\exp(-ikx)\,dx \tag{3.5.1}$$

where now k takes on all values. We regain the original function by doing the inverse Fourier transform.

$$y(x) = (1/2\pi)\int_{-\infty}^{\infty} \tilde{y}(k)\exp(ikx)\,dk \qquad (3.5.2),$$

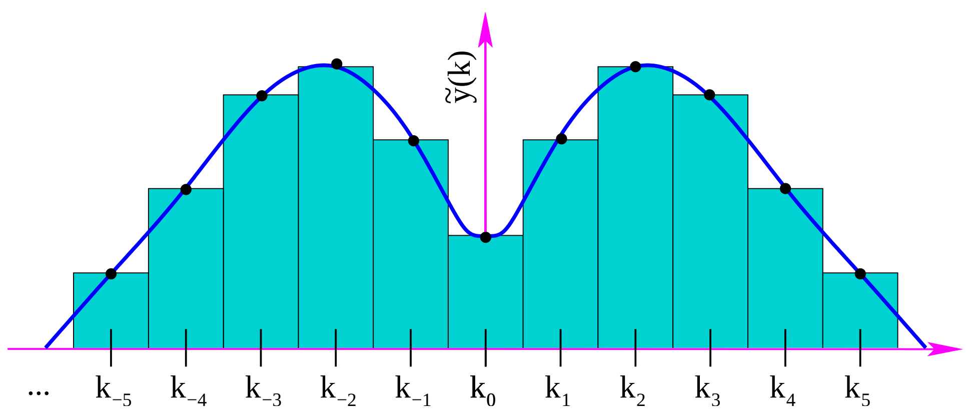


Figure 4.6, Approximating the integral as a sum. By approximating the integral $\int \tilde{y}(k)\exp(-ikx)\,dk$ as a sum over the equally spaced points k_m, $\sum_m \tilde{y}(k)\exp(-ik_m x)\Delta k$, we can connect the formula for the Fourier transform to the formula for the Fourier series.

(a) **Series → Transform.** Let $y(x)$ be a smooth function which is zero outside $(0, L)$. By what constant do you need to multiply the Fourier series coefficient $\tilde{y}_m$ in equation (1.2.3) to get the Fourier transform $\tilde{y}(k_m)$ in (3.5.1)? Approximating the Fourier transform integral (3.5.1) as a sum (as shown in Figure 4.6), use the Fourier series formulas (1.2.2) and (1.2.3) to explain or derive the factor $(1/2\pi)$ in equation (3.5.2).

As we take $L \to \infty$ the spacing between the points k_m, $2\pi/L$, gets smaller and smaller, and the approximation of the integral as a sum gets better and better.

There is a remarkably fast numerical method, called the Fast Fourier transform. It starts with N equally spaced data points y_ℓ, and returns a new set of complex numbers $\tilde{y}_m^{FFT}$:

$$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N). \qquad (4.6.1)$$

(b) **FFT → Series.** We can use the FFT to give an approximation to the Fourier series. Let $y_\ell = y(x_\ell)$ where $x_\ell = \ell(L/N) = \ell(\Delta x)$. As in part (a), approximate the Fourier series integral (1.2.3) above as sum over y_ℓ, $(1/L)\sum_{\ell=0}^{N-1} y(x_\ell)\exp(-ik_m x_\ell)\Delta x$. For small positive m, give the constant relating $\tilde{y}_m^{FFT}$ to the Fourier series coefficient $\tilde{y}_m$. The Fourier series is defined for both positive and negative m, where the FFT gives only positive m. Show that $\tilde{y}_m^{FFT} = \tilde{y}_{m+N}^{FFT}$, and then argue how you can use this to get the Fourier series coefficients for negative m by looking at the FFT near the end of the list,

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 5, due Monday Sept. 28
Latest revision: December 18, 2003, 3:15 pm

Computer Labs

Fourier Lab I, Monday evening 9/22 and Tuesday afternoon 9/23, Rock B3 (hidden around the corner in the basement).

Prelim I

Prelim I is scheduled next week, Friday October 3. The content will focus on the homework, the experimental lab *Standing Waves* and the two Fourier labs. There will be multiple-choice questions (no partial credit) and one or perhaps two longer multiple-part essay questions. Next Monday, instead of a problem set, I will pass out copies of last year's Prelim I for you to use while studying.

Reading

Elmore & Heald, sections 5.1-5.3, 5.10, 12.3/5

Feynman, section I.48-5/6, I.49-3/5, I.50-5/6, I.51 1/2, I.52, II.25-10.

Problems

Elmore & Heald, page 142, problem 5.2.3 (squeaky voices with Helium).

(5.1) Deriving the Wave Equation: Symmetries *vs.* Free Body Diagram

A thin horizontal string of density λ and tension τ is vibrating inside a viscous fluid. It is subject to a transverse viscous force $b\,\partial\eta/\partial t$ per unit length so as to oppose the transverse motion of the string. In addition, it is subject to an external gravitational force.

(a) **Free Body Diagram Method.** Generalize the derivation of E&H equation (1.1.2) to incorporate these effects of viscosity and gravity. Make sure to draw the appropriate free body diagram for the chunk of string with two tension forces, the viscous force, and the force due to gravity.

Part (a) could have asked you to "Incorporate into the wave equation the leading order terms breaking time-reversal invariance and invariance under changing the sign of the order parameter. Give a possible physical origin for each term." You've already solved this problem in part (a): part (b) asks you to translate the answer into the language of modern condensed-matter physics:

(b) **Symmetry Method.** Which term, gravity or friction, breaks time-reversal invariance? Which term breaks invariance under changing the sign of the order parameter?

(5.2) Deriving New Laws.

The evolution of a physical system is described by a field Ξ, obeying a partial differential equation

$$\partial\Xi/\partial t = A\,\partial\Xi/\partial x. \tag{5.2.1}$$

(a) **Symmetries.**

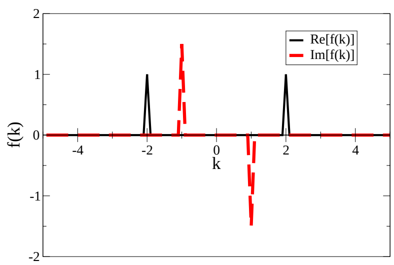
Give the letters corresponding to ALL the symmetries that this physical system appears to have:

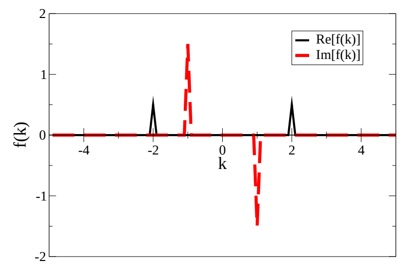
(A) Spatial inversion ($x \to -x$).
(B) Time reversal symmetry ($t \to -t$).
(C) Order parameter inversion $\Xi \to -\Xi$).
(D) Homogeneity in space ($x \to x + \Delta$).
(E) Time translational invariance ($t \to t + \Delta$).
(F) Order parameter shift invariance ($\Xi \to \Xi + \Delta$).

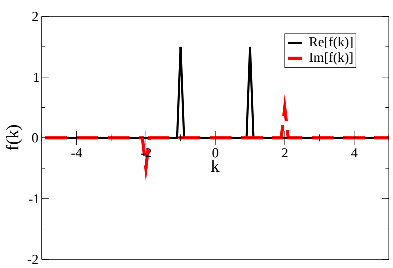
(b) **Traveling Waves.** Show that our equation $\partial\Xi/\partial t = A\,\partial\Xi/\partial x$ has a traveling wave solution. If $A > 0$, which directions can the waves move?

(5.3) Fourier Series.

Which picture represents the Fourier series associated with the function $f(x) = 3\sin(x) + \cos(2x)$? (The solid line is the real part, the dashed line is the imaginary part.)

(A)


(B)


(C)


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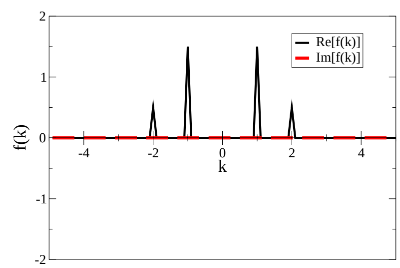
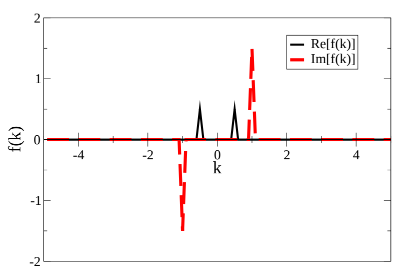
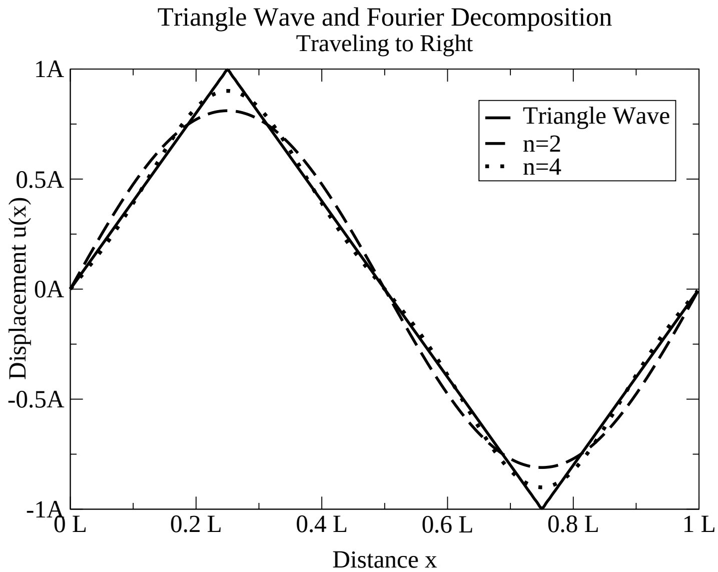


(E)


(5.4) Sawtooth Wave.



A sound wave generator generates a triangular pressure air wave moving toward the right down a hollow tube, as shown in the figure above. The triangles repeat forever with wavelength L. The maximum displacement of the wave is A, the velocity of sound is v, and the bulk modulus for air is B.

(a) What is the intensity (power per unit area) traveling down the tube?

The figure shows the Fourier series for our wave truncated at $n = \pm 2$ and $n = \pm 4$.

(b) We now want to decompose this intensity into different frequencies. Give the time average intensity I_n^{av} of a single traveling plane wave of wave vector k_n and amplitude a_n, $u_n(x, t) = a_n \sin(k_n(x - vt))$? (Leave your answer in terms of a_n and k_n.)

The Fourier series for the displacement of the wave is

$$u(x) = \sum_{n=0}^{\infty} a_n \sin(k_n(x - vt))$$

with $k_n = 2\pi n/L$. The Fourier coefficients are $a_n = 0$ for n even, and for n odd are

$$a_n = (-1)^{(n-1)/2} 8A/(\pi^2 n^2).$$

(c) Verify explicitly that the sum of the intensities per frequency channel n you calculated in part (b) equals the total intensity you calculated in part (a). You'll need the formula $\pi^2/8 = 1 + 1/3^2 + 1/5^2 + 1/7^2 + \dots$

This is Feynman's energy theorem, section I.50-5: the energy of the sum of different Fourier waves is the sum of the energies of the individual waves. This is why we can talk about the power spectrum of a wave: you can think of the power at different frequencies as being independent of one another.

(5.5) Pythag: Group velocity, phase velocity, and dispersion.

Start up Pythag. Choose *Packet* forcing on the left-hand side: this yanks on the left sinusoidally with a frequency $\Omega = 300$ radians per second, with an amplitude given by a Gaussian pulse of FWHM 0.04 seconds Hit *Initialize* and *Run*, and watch the packet bounce back and forth. As is usual with the wave equation, the pulse propagates without changing in shape. This is only true, however, so long as the pulse does not change much on the length scale given by the distance between points δx on the numerical string.

Open the *Configure* menu. Change Ω to 800 and $FWHM$ to 0.015. To slow down the pulse, lower *graph time skip* to one. You should now see a pulse which changes shape as it moves.

(a) Is the group velocity faster or slower than the phase velocity? This is easiest to see by looking at the pulse early on, before it stretches out: do the peaks within the wave of the carrier frequency move forward faster or slower than the pulse as a whole?

After several passes across the window, you should see a broad pulse, which has longer waves on one side than the other.

(b) Does the leading edge have longer or shorter wavelength than the trailing portion of the packet? Which wavelengths move faster, the long wavelengths or the short ones?

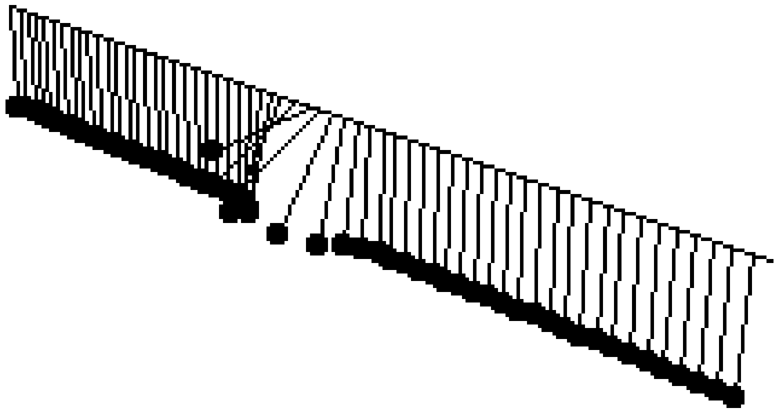
This is called *chirping*. Try making a sound that goes up in pitch at the end: what does it sound like?

(c) Do these two answers agree with what you found for the dispersion relation in problem set 4?

Now change the number of string pieces (chunks) to 999 (the largest value allowed), and change the graph time skip back to 20.

(d) Does the dispersion go away when you reduce the spacing δx in this way?

(5.6) Sine-Gordon Dispersion Relation.



An array of pendula connected by springs, in the continuum limit, obeys the Sine-Gordon equation

$$\partial^2\phi/\partial t^2 = A\partial^2\phi/\partial x^2 - B\sin(\phi).$$

with $\phi(x) = 0$ corresponds to the pendulum at position x along the array pointing downward. What is the dispersion relation $\omega(k)$ for small oscillations in this equation?

(A) $\omega(k) = \left(-B \pm \sqrt{B^2 - 4Ak^2}\right)/2A$

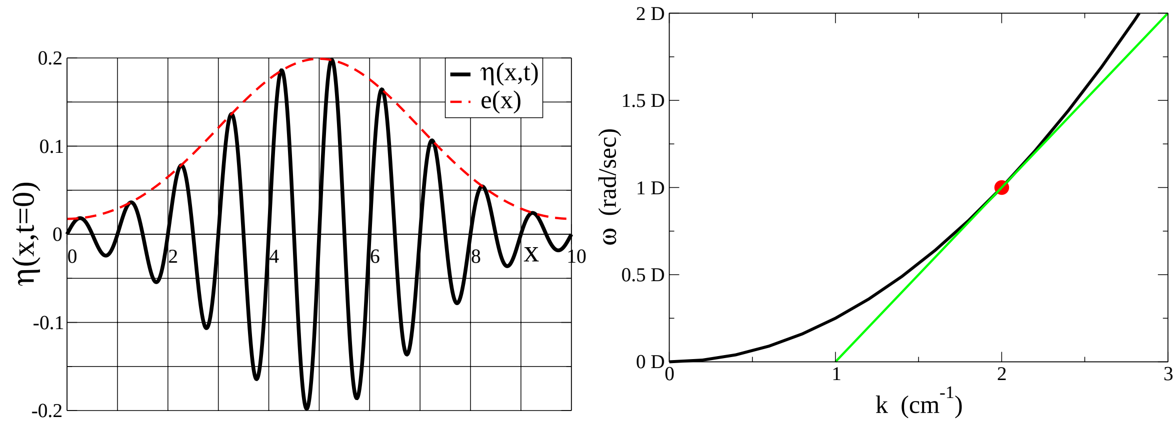
(B) $\omega(k) = \sqrt{Ak^2 - B\sin(\phi)}$

(C) $\omega(k) = \sqrt{Ak^2 - B}$

(D) $\omega(k) = \sqrt{Ak^2 + B}$

(E) $\omega(k) = \sqrt{Ak^2 + B\sin(\phi)}$

(5.7) Group and Phase Velocities.



A wave packet in an unusual optical fiber is traveling to the right. The wave packet has an envelope of shape $e(x)$ at time $t = 0$, and has a carrier wave with nodes at $x = 0.5$cm, $x = 1$cm, $x = 1.5$cm, ... as shown. The dispersion relation for the glass in this fiber is $\omega(k) = Dk^2$, as shown.* Which of the following formulas most closely describes $\eta(x,t)$, ignoring the slow spreading of the wave packet?

(A) $\eta(x,t) \approx e(x - 4\pi Dt)\sin(2\pi(x - 2\pi Dt))$.

(B) $\eta(x,t) \approx e(x - 2\pi Dt)\sin(2\pi(x - 1\pi Dt))$.

(C) $\eta(x,t) \approx e(x - 2\pi Dt)\sin(2\pi(x - 2\pi Dt))$.

(D) $\eta(x,t) \approx e(x - 2\pi Dt)\sin(2\pi(x - 4\pi Dt))$.

(E) $\eta(x,t) \approx e(x - 1\pi Dt)\sin(2\pi(x - 2\pi Dt))$.

(F) $\eta(x,t) \approx \exp(ikx - \omega t)$.

* It's not useful for this problem, but this is the dispersion relation for the wave equation $\partial^2\eta/\partial t^2 = -D\partial^4\eta/\partial x^4$.

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 6, due Wednesday Oct. 15
Latest revision: December 18, 2003, 3:15 pm

Reading

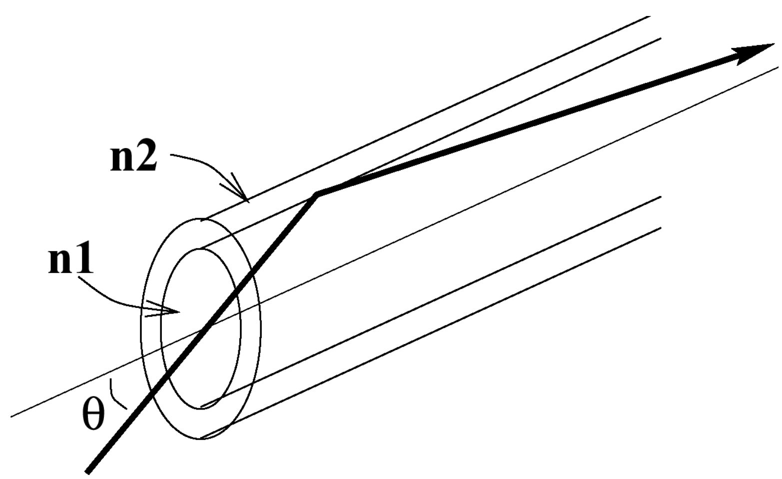
Elmore & Heald, sections 5.5-5.7 (Waves in 3D Fluids)
Feynman, section I.26 (Least Time), I.27 (Geometrical Optics), I.28 (Dipole Radiator), I.29 (Interference), II.20-4 (Spherical Waves)

Experimental Lab II

Microwaves and Optics, Monday evening 10/6 and Tuesday afternoon 10/7, Rock B26 and B30.

Problems

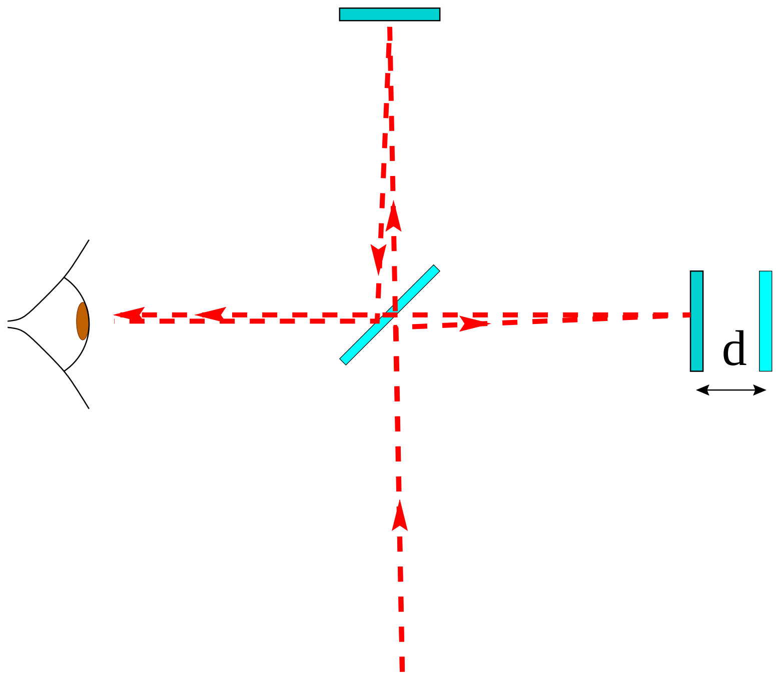
(6.1) Optical Fibers and Total Internal Reflection. An optical fiber consists of a glass core (index of refraction n_1) surrounded by a coating (index of refraction $n_2 < n_1$). Suppose a beam of light enters from air obliquely at an angle θ with the fiber axis as shown in the figure below.



(a) Show that the greatest possible value of θ for which a ray can be propagated down the fiber without leaking out is given by $\theta = \sin^{-1}(n_1^2 - n_2^2)^{1/2}$. Assuming that the glass and coating indices of refraction are 1.55 and 1.50, respectively, calculate θ_{max}.

(b) What would the critical angle be if the outer layer of glass were not there?

(6.2) Michelson Interferometer.



As one of the mirrors of a Michelson interferometer is moved through a distance d of 0.163 mm, 500 bright fringes move across the field of view. What is the wavelength of the light illuminating the mirrors of the interferometer?

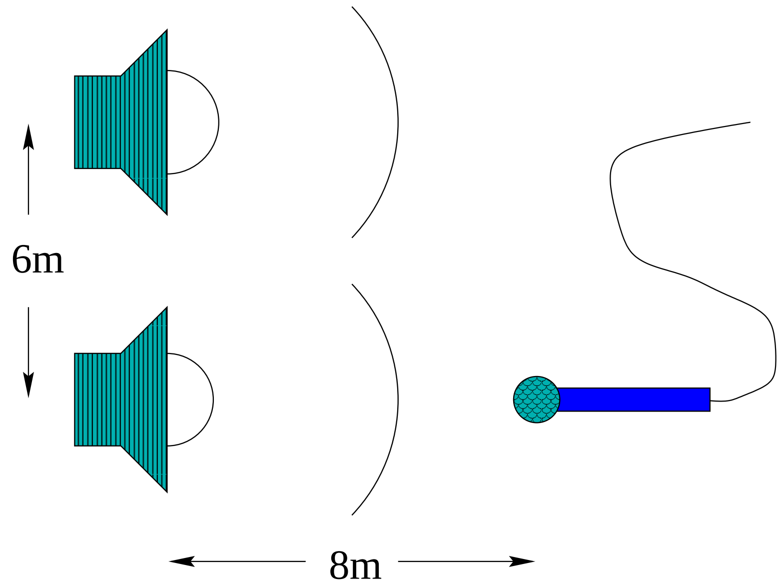
(6.3) Reflectionless Coatings. A string has three segments, the first of densities $\mu_1 = 0.1$kg/m, the second a short segment of density $\mu_2 = 0.05$ kg/m, followed by a segment of density $\mu_3 = 0.025$. The string is under tension $\tau = 160$N. Sinusoidal waves of frequency $\omega = 300$rad/s impinge from the left.

(a) How long should the middle segment be to minimize the reflection?

This is an example of a *reflectionless coating*. Your glasses may have such a coating, designed to reduce the reflections of light from their surface. (It's much more work to design one that works at all wavelengths...)

(b) Check your answer to part (a) with Pythag. The REFJUMP preset should set things up properly: change X12 or X23 on the *Configure* menu to change the length of the middle, red segment. Test to make sure your answer does indeed give less reflection than longer or shorter segments. (Zooming in on the reflected pulse on the $y(t)$ plots makes it easy to measure the amplitude to high accuracy.)

(6.4) Sound Wave Interference.



Suppose there are two loudspeakers emitting spherical sound waves, a distance $d = 6m$ apart along the y axis (at $x = 0, y = \pm 3$m). The sources emit sound at the same frequency, and are in phase. Consider the point B at $x = 8$m, $y = 3$m, directly in front of one of the loudspeakers. If the wavelength of sound is two meters, is there constructive or destructive interference? How about a wavelength of 4m? Check these qualitatively using the program Huygens, which you downloaded along with Pythag for an earlier assignment. (Put "X Screen" to 8 m, d to 6 m, and the screen size to something sensible.) Is the intensity exactly zero for the case of destructive interference?* (Zoom in on the graph with the right mouse button.) Why not? What relative intensity I_{+3}/I_{-3} of the sources would produce zero sound level at B for the destructive case, for point sources of sound?

(6.5) Double Thin Slit. A double slit with slit separation d is illuminated by coherent light of wavelength λ. The lower slit is covered by a piece of glass of thickness t and refractive index $n = 1.3$. An interference pattern is observed on a screen a distance $D >> d$ away. (a) At what angle θ will the principle $m = 0$ maximum of the interference pattern be? (You may assume that θ is small.) (b) At what minimum thickness will the interference pattern show destructive interference at $\theta = 0$?

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* Huygens simulates a source which is a thin slit, rather than a point source, so the decay of amplitude with distance is different than the one for the analytical portion of this problem.

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 7, due Monday Oct. 20
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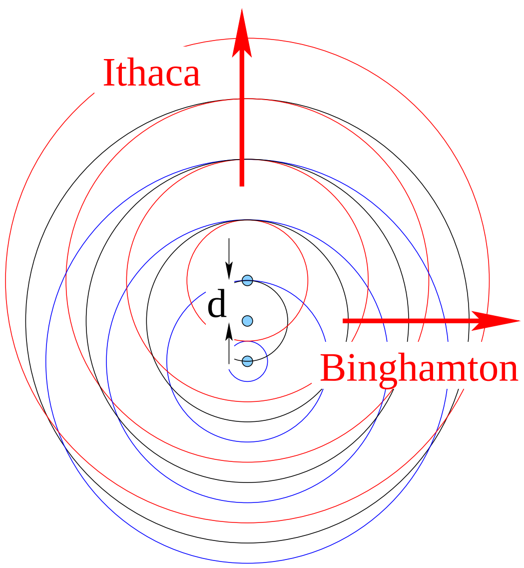
Reading

Feynman, sections I.30 (Diffraction), II.30 (Crystals), and II.31 (Tensors)

Problems

(7.1) Antenna Array



A radio station south of Ithaca and west of Binghamton transmits with a carrier wave of wavelength λ. They use three antennas spaced equally along a north-south direction, separated by a distance $d = 2/3\,\lambda$. The antennas are close together compared to the distance to either city. The radio station cleverly delays the signal in the antennas, so as to add a phase shift $\phi = 2\pi/3$ in the carrier wave between neighboring antennas. In particular, if we number the three antennas from south to north as $\{-1, 0, 1\}$, the radio signal is given by

$$A(r_0)\,(\exp[i(\omega t - kr_1 + \phi)] + \exp[i(\omega t - kr_0)] + \exp[i(\omega t - kr_{-1} - \phi)]$$

What is the ratio of the average intensity I_{av} of the radio signal in Ithaca, compared to I_0, which one would find with only one antenna transmitting? In Binghamton?

(A) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 3$.

(B) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 1$.

(C) Ithaca: $I_{\text{av}}/I_0 = 9$; Binghamton $I_{\text{av}}/I_0 = 0$.

(D) Ithaca: $I_{\text{av}}/I_0 = 3$; Binghamton $I_{\text{av}}/I_0 = 1$.

(E) Ithaca: $I_{\text{av}}/I_0 = 3$; Binghamton $I_{\text{av}}/I_0 = 0$.

(7.2) Interference

A coherent laser beam impinges on a slit of width a. An intensity pattern is viewed on a distant screen: the center has intensity I_0 and the peak width (distance between the nearest minima) is ΔY. The slit is broadened to $2a$. What is the new intensity $I_{doubled}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin\theta \approx \theta$.

(A) $I' = 4I_0$, $\Delta Y' = \Delta Y/2$.

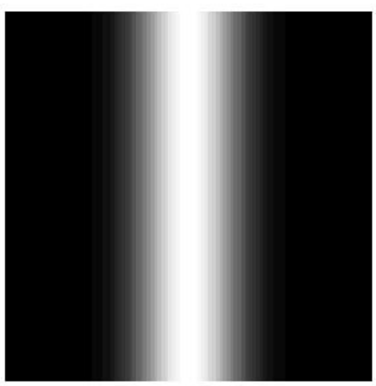
(B) $I' = 2I_0$, $\Delta Y' = \Delta Y/2$.

(C) $I' = 2I_0$, $\Delta Y' = \Delta Y/4$.

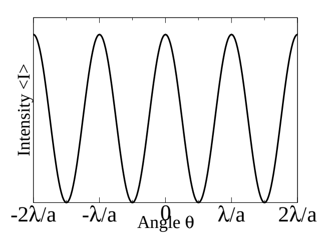
(D) $I' = 4I_0$, $\Delta Y' = 2\Delta Y$.

(E) $I' = 2I_0$, $\Delta Y' = 2\Delta Y$.

(7.3) Diffraction and Fourier Optics.



A glass slide is coated with soot, blocking all light except for a thin stripe across the center where the soot has been rubbed off. A laser beam of wavelength λ is aimed at the glass: the beam width is large compared to the slit width. A careful measurement of the light transmission immediately outside the glass shows that amplitude of the light has a Gaussian profile: it varies with x as $\exp(-2x^2/a^2)$, where a is the width (standard deviation) of the transparent strip and x is the distance from the center of the strip. Which of the following intensity patterns will be observed on a distant screen? (You may assume small angles.)

(A)


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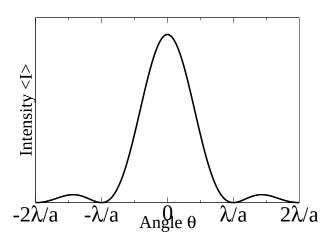
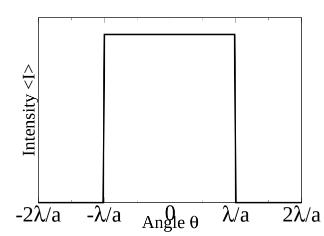


(C)


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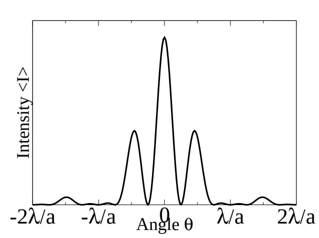
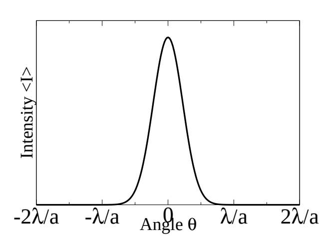


(E)


(7.4) Introduction to Tensors. In the next few weeks, we'll make heavy use of *tensors*. Tensors are a generalization of vectors and matrices: vectors v_i are one-index tensors, matrices M_{ij} are two-index tensors, and we'll be making use of three and four-index tensors like $c_{ijk\ell}$ in our discussions of elasticity in solids. Just as for a vector or a matrix, a tensor $c_{ijk\ell}$ is a multidimensional array of real numbers, one for each choice of i, j, k, and ℓ ranging from one to three.

(a) How many different real numbers are needed to specify a general four-index tensor?

In this problem, we introduce two particularly useful and important tensors. One is the Kronecker delta function δ_{ij}, which is one if $i = j$ and zero if $i \neq j$.

$$\delta_{ij} = 1 \quad \text{if } i = j \qquad \delta_{ij} = 0 \quad \text{if } i \neq j \tag{7.4.1}$$

The other is the Levi-Civita symbol, or totally antisymmetric tensor, ϵ_{ijk}. It is defined by its value for $i = 1$, $j = 2$, $k = 3$,

$$\epsilon_{123} = 1 \tag{7.4.2}$$

and its antisymmetry property: it changes sign whenever two indices are permuted:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \tag{7.4.3}$$

It's easy to see that ϵ_{ijk} gives $+1$ if $\{ijk\}$ is an even permutation of $\{123\}$, -1 if it is an odd permutation, and zero if any two indices agree.

(b) Using equations (7.4.2) and (7.4.3), show that (specifically) $\epsilon_{223} = 0$; show also that $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$.

(c) Write out δ_{ij} as a matrix, with i labeling the row and j the column. What do you usually call this matrix? Write out ϵ_{ijk} as three matrices ϵ_{1jk}, ϵ_{2jk}, and ϵ_{3jk}, with i labeling the matrix, j the row and k the column. (We do not usually write out tensors in this way.)

One of the most common things we do to tensors is taking *outer products* and/or *contracting* them. The outer product of two tensors a_{ij} and $b_{k\ell}$, for example, is a tensor with four indices given by the product of the two: $d_{ijk\ell} = a_{ij}b_{k\ell}$. Contraction is done by setting two indices of a tensor (or an outer product of tensors) equal, and summing over all values of that repeated index: the new tensor has two fewer indices after contraction.

A familiar examples of a contraction is taking the trace of a matrix: $tr(M) = \sum_{i=1}^{3} M_{ii}$. Three familiar examples of taking outer products and then contracting are the dot product of two vectors, $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{3} v_i w_j$, applying matrices to vectors $(M\mathbf{v})_i = \sum_{j=1}^{3} M_{ij} v_j$, and multiplying matrices $(MN)_{ik} = \sum_{j=1}^{3} M_{ij} N_{jk}$.

You notice that there are a lot of sums $\sum_{j=1}^{3}$ in the formulas above. In physics, we often make use of the **Einstein convention**, where summation (contraction) over repeated indices is implied. Hence if we write a_{iij}, the convention implies that we really meant the one-index tensor resulting from summing over i, $\sum_{i=1}^{3} a_{iij}$.

(d) Write the trace, dot product, matrix operating on a vector, and matrix multiplication examples above using the Einstein convention.

(e) Give arguments for the following formulas involving δ_{ij} and ϵ_{ijk}. We use the Einstein convention.

$$\delta_{ii} = 3$$

(Easy.)

$$\epsilon_{ijk}\delta_{jk} = 0$$

(Consider two kinds of terms: $j = k$ and $j \neq k$.)

$$\epsilon_{ijk}\epsilon_{ijk} = 6$$

(Show that this is the sum of the squares of all of the elements of the tensor. How many non-zero elements are there?)

$$\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$$

(Show that the left–hand–side is zero if $k \neq \ell$. Then compute it for $k = \ell = 3$, and argue from there.)

$$\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$$

(Show the left–hand–side is zero except in the two cases ($i = k$ and $j = \ell$) and ($i = \ell$ and $j = k$). Then find the sign for the two cases.)

(f) Write the cross product of two vectors $\mathbf{v}$ and $\mathbf{w}$ as the outer product contracted twice with the totally antisymmetric tensor.

(g) (Optional: for the ambitious and inspired.) The determinant of a matrix M is antisymmetric under interchange of any two rows or columns, and the determinant of the identity matrix is one. Argue (without doing messy calculations) that these are also true of the formula

$$\det M = (1/6)\, \epsilon_{ijk}\epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn}.$$

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 8, due Monday Oct. 27
Latest revision: December 18, 2003, 3:15 pm

Reading

Feynman, section I.30 (Diffraction), II.30 (Crystals), II.31 (Tensors), II.38-1/2 (Elasticity), II.39.1/5 (Strain and Elasticity)

Possibly also useful: Elmore & Heald, sections 3.1-3.3, 7.4

Prelim II

Prelim II is tentatively scheduled for Wednesday November 5, pending discussion about timing with the class. Prelim II will cover the higher-dimensional wave equations, interference and diffraction, tensors, elasticity theory, elastic waves, and electromagnetic waves. It will potentially include questions from the experimental lab *Microwaves and Optics*. It will be a similar format to the last exam.

Problems

(8.1) Diffraction Grating. A 10 cm wide diffraction grating with 10000 slits is used to measure the wavelengths emitted by hot hydrogen gas. (a) At what angles θ in the first order spectrum do we expect to find the two violet lines of wavelengths 434 and 410nm? (b) Same question but for second order.

(8.2) Thick Slits and Windows. Start up Huygens. Set the "Number of Slits" to one and the width a of the slit to 5 m. Notice the single-slit diffraction pattern on the right. Notice that the waves on the left look much like you'd expect for light coming in a window: light traveling along straight lines. How do we reconcile these two pictures?
Bring the screen in closer: try "X Screen" at 10 and 1 meter. (If you get too close, you'll begin to see my numerical method for generating the slit.) Now vary the wavelength. How much farther does the intensity look "window-like" at $\lambda = 0.35$ than for $\lambda = 0.7$? Finally, vary the slit width a. With $\lambda = 0.7$, what distance to the screen for $a = 2.5$ looks the same shape as 10m for $a = 5$?
Explain why you can hear around corners, but you can't see around corners.

(8.3) Double Thick Slit. Start up Huygens. Set d to 8m, and a to 2m. You should see a complicated interference pattern. Now set the number of slits to one. Is the single slit pattern the envelope of the double thick-slit pattern? Set the number of slits back to two, and set a to zero. Is the thin-slit pattern like the carrier wave?
Show that the intensity for a double-slit with distance d between the centers and width a for each slit is the product of the single-slit diffraction pattern of width a and the double thin-slit diffraction pattern. (Hint: Remember that the amplitude of the wave at the screen is proportional to the Fourier transform of the slit-opening function $E(\theta) = Re\left(Ae^{i\omega t}\tilde{f}(k\sin\theta)\right)$. You can find how the amplitude from the upper slit changes as it is translated upward by $d/2$ by using the properties of Fourier transforms under translation.)

(8.4) Interference A coherent laser beam impinges on a slit of width a. An intensity pattern is viewed on a distant screen: the center has intensity I_0 and the peak width (distance between the nearest minima) is ΔY. The slit is broadened to $2a$. What is the new intensity $I_{doubled}$ and peak minimum separation $\Delta Y'$? You may assume that the angles are small, so $\sin\theta \approx \theta$.

(A) $I' = 4I_0$, $\Delta Y' = \Delta Y/2$.

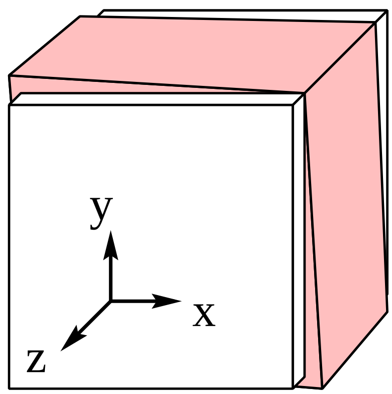
(B) $I' = 2I_0$, $\Delta Y' = \Delta Y/2$.

(C) $I' = 2I_0$, $\Delta Y' = \Delta Y/4$.

(D) $I' = 4I_0$, $\Delta Y' = 2\Delta Y$.

(E) $I' = 2I_0$, $\Delta Y' = 2\Delta Y$.

(8.5) Traction-free boundary condition.



An isotropic elastic medium is strained as shown above: it is compressed and stretched along different axes. The stress tensor is

$$\sigma_{ij} = \begin{pmatrix} a & -a & 0 \\ -a & a & 0 \\ 0 & 0 & -2a \end{pmatrix}.$$

The medium has a flat free surface perpendicular to the axis $\hat{\mathbf{n}}$. A free surface is a surface on which there is no traction, or forces, applied: for example, a surface exposed to vacuum (or approximately, to air). Knowing the stress tensor above, in which direction $\hat{\mathbf{n}}$ could the surface normal point?

(A) $\hat{\mathbf{n}} = (1, 0, 0)$

(B) $\hat{\mathbf{n}} = (0, 1, 0)$

(C) $\hat{\mathbf{n}} = (0, 0, 1)$

(D) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 0)$

(E) $\hat{\mathbf{n}} = (1/\sqrt{2}, -1/\sqrt{2}, 0)$

(F) $\hat{\mathbf{n}} = (1/\sqrt{2}, 1/\sqrt{2}, 1/\sqrt{2})$

Draw $\hat{\mathbf{n}}$ and the strained cube, and convince yourself that the surface perpendicular to $\hat{n}$ would be free.

Related formulæ: $F_i/A = \sigma_{ij}\hat{\mathbf{n}}_j$, $F_i = \partial_j \sigma_{ij}$, $\sigma_{ij} = c_{ijkl}\epsilon_{kl} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$

(8.6) The Power of Tensors. Remember from last week the definitions of the two most important tensors: the Kronecker delta function $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$, and the totally antisymmetric tensor $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$ with $\epsilon_{123} = 1$. Remember the identities that you proved: $\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, and $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$. Notice that one can conveniently use tensor notation to write the gradient $(\nabla\psi)_i = \partial_i\psi$, divergence $\nabla \cdot \mathbf{a} = \partial_i a_i$, and curl $(\nabla \times \mathbf{a})_i = \epsilon_{ijk}\partial_j a_k$.

Use these formulas to prove the following vector identities (listed in the front of Jackson, *Classical Electrodynamics*):

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times \nabla\psi = 0$$

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla \cdot (\psi\mathbf{a}) = \mathbf{a} \cdot \nabla\psi + \psi\nabla \cdot \mathbf{a}$$

$$\nabla \times (\psi\mathbf{a}) = \nabla\psi \times \mathbf{a} + \psi\nabla \times \mathbf{a}$$

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$$

HInt: You'll need to use the fact that the second derivative doesn't depend upon the order of variables ($\partial_i\partial_j f = \partial_j\partial_i f$) and the product rule ($\partial_i(f\,g) = (\partial_i f)g + f(\partial_i g)$).

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 9, due Monday Nov. 3
Latest revision: December 18, 2003, 3:14 pm

Reading

Feynman II.39.1/5 (Strain and Elasticity), I.51.3/4 (Waves in Solids and Surface Waves), II.39-3/4 (Elastic Motion), II.19 (The Principle of Least Action: not for credit), II.20 (Solutions of Maxwell's Equations in Free Space), and II.32 (Refractive Index of Dense Materials).

Possibly also useful: Elmore & Heald, sections 7.4, 7.5, 7.6, 8.1-8.5; don't worry about the obsolete notation of dyadics and stuff.

Prelim II

Prelim II is tentatively scheduled for Wednesday November 5, pending discussion about timing with the class. Prelim II will cover the higher-dimensional wave equations, interference and diffraction, tensors, elasticity theory, elastic waves, and electromagnetic waves. It will potentially include questions from the experimental lab *Microwaves and Optics*. It will be a similar format to the last exam.

Experimental Lab III

Interference and Diffraction, Monday evening 11/10 and Tuesday afternoon 11/11, Rock B26 and B30.

Problems

(9.1) Strain fields at large rotations. Show for a rotation about the z axis by an angle θ that the gradient of the displacement field $\partial_j u_i$ is

$$\vec{\nabla}\vec{u} = \begin{pmatrix} \cos(\theta)-1 & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta)-1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Calculate the strain field $\epsilon_{ij}^{\text{approx}} = (1/2)(\partial_i u_j + \partial_j u_i)$ and show that it is not small. How large a rotation would give a 1% strain component to the strain tensor (and hence lead to plastic deformation)? Now calculate the true strain matrix including the "geometric nonlinearity" $\epsilon_{ij} = (1/2)(\partial_i u_j + \partial_j u_i + \partial_i u_k \partial_j u_k)$ and show that it is zero.

(9.2) Elastic Moduli.

In an isotropic material, only two elastic moduli are independent: all others can be written in terms of them. The tensor relation between strain and stress is most nicely written in terms of the two Lamé elastic constants λ and μ:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}. \quad (8.6.1)$$

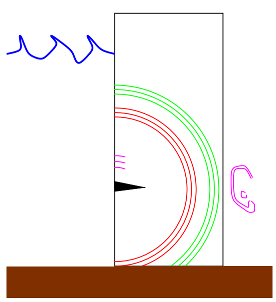
The constant μ is just the shear modulus (Feynman eqn. II.38.14); the constant $\lambda = B - 2\mu/3$, where B is the bulk modulus (Feynman's K, eqn. II.38.9).

(a) Show that bulk compression has three non-zero components: $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \Delta V/(3V)$. Knowing that the stress $\sigma_{xx} = B\Delta V/V$, check the formula for the bulk modulus given above.

If we put a force per unit area $\sigma_{xx} = F/A$ stretching our material in the x-direction, and we don't constrain the sides (so $\sigma_{ij} = 0$ except along x, $i = j = 1$), the material will stretch by an amount $\Delta L/L = (1/Y)F/A$, where Y is the Young's modulus. It also compresses in the y and z directions by Poisson's ratio* ν times the extension along the x direction, so the width and height change by $\Delta W/W = \Delta H/H = -\nu\Delta L/L$.

(b) Write the stress tensor σ_{ij} and the strain tensor ε_{kl} for this problem, as 3×3 matrices.

(c) Use your answers to part (b) and the equation (8.6.1) relating stress and strain for isotropic materials, solve for Young's modulus Y and Poisson's ratio ν in terms of λ and μ.



(9.3) Elastic Waves.

A small crack starts on the inside of a concrete dam, generating acoustic waves of all polarizations with wavelengths much shorter than the thickness D of the dam. An acoustical detector is positioned outside the dam directly opposite to the crack. The concrete can be assumed to be an isotropic medium with positive elastic constants λ and μ. What signal is expected in the acoustical detector?

(A) A transverse sound pulse, followed by a longitudinal sound pulse.
(B) A longitudinal sound pulse, followed by a transverse sound pulse.
(C) A transverse sound pulse only: sound is a transverse wave.
(D) A longitudinal sound pulse only: the transverse sound component will travel along the length and width of the dam, not across the thickness.
(E) A sound pulse after a time $t = D/\sqrt{Y/\rho}$, where Y is the Young's modulus of concrete.

Related formulæ: $\rho\partial^2 u_i/\partial t^2 = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$. With $\nabla\cdot\mathbf{u}_T = 0$, $c_T = \sqrt{\mu/\rho}$; with $\nabla\times\mathbf{u}_L = 0$, $c_L = \sqrt{(\lambda+2\mu)/\rho}$

* We use ν for Poisson's ratio, as the engineers do, reserving σ for the stress tensor.

(9.4) Tensor Notation Review. Suppose $\mathbf{B} = \nabla\times\mathbf{A}$. Which of the following are correct formulas for $\mathbf{B}^2$? (For example, the energy contained in a magnetic field is $\mathbf{B}^2/8\pi$.)

(A) $\varepsilon_{ijk}\partial_j A_k \varepsilon_{i\ell m}\partial_\ell A_m$.
(B) $(\delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell})(\partial_j A_k)(\partial_\ell A_m)$.
(C) $(\partial_j A_k)^2 - (\partial_j A_k \partial_k A_j)$.
(D) All of the above.
(E) None of the above.

(9.5) Elastic Traveling Wave. An isotropic elastic medium with density ρ and moduli λ and μ fills the half space $x > 0$. The boundary of this medium is wiggled with displacement field

$$\mathbf{u}(0, y, z) = (f(t), g(t), h(t)),$$

generating an elastic wave travelling to the right (positive x direction).
What is the displacement $\mathbf{u}(x, y, z, t)$ for $x > 0$?

(A) $\mathbf{u}(x, y, z, t) = (0, g(t - x/c), h(g - x/c))$.
(B) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{(\lambda + 2\mu)/\rho}), g(t - x/\sqrt{\mu/\rho}), h(t - x/\sqrt{\mu/\rho}))$.
(C) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - x/\sqrt{(\lambda + 2\mu)/\rho}), h(t - x/\sqrt{(\lambda + 2\mu)/\rho}))$.
(D) $\mathbf{u}(x, y, z, t) = (f(x - \sqrt{\mu/\rho}\,t), g(x - \sqrt{(\lambda + 2\mu)/\rho}\,t), h(x - \sqrt{(\lambda + 2\mu)/\rho}\,t))$.
(E) $\mathbf{u}(x, y, z, t) = (f(t - x/\sqrt{\mu/\rho}), g(t - y/\sqrt{(\lambda + 2\mu)/\rho}), h(t - z/\sqrt{(\lambda + 2\mu)/\rho}))$.

(9.6) Waves on a Thin Wire. A plane wave of wave vector k passes along the $\hat{x}$ direction through a thin wire of radius W. The wire width W is thin compared to the wavelength, so $kW << 1$. The material making up the wire is isotropic, with elastic moduli λ and μ. The wave at $t = 0$ is approximately given by the real part of

$$\mathbf{u} = Ae^{i(kx-\omega t)}\left(1 - \nu\frac{k^2(y^2 + z^2)}{2}, -iky\nu, -ikz\nu\right)$$

where we use the engineering notation ν for Poisson's ratio $\nu = \lambda/2(\mu + \lambda)$.* This formula is correct up to terms of order k^3W^3. The wave is primarily longitudinal, for small k (the y and z components of $\mathbf{u}$ are smaller by a factor of kW than the x component). The wave is basically stretching and compressing the wire along the $\hat{x}$ direction, with a small correction.

(a) Ignoring for the moment the term proportional to k^2, show that the y and z components are just what one would expect from Poisson's ratio applied to the amount the wire is stretched along the x direction.

The k^2 term took me a long time to figure out the first year I taught this. I don't have a simple explanation for it, but without keeping it you get the wrong sound velocity even as $k \to 0$.

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* Feynman and E&H use σ for Poisson's ratio, which we use for the stress tensor.

(a) Compute the strain tensor $\varepsilon(x, y, z, t)$ for this displacement field, ignoring the geometric nonlinearity. Write it out as a 3×3 matrix.

(b) The wire is isotropic, with elastic moduli λ and μ. Write the stress tensor for the wire as a 3×3 matrix.

(c) (Not for credit: gluttons for punishment only.) Check that this displacement field satisfies Newton's law

$$\rho\partial^2 u_i/\partial t^2 = \partial_j \sigma_{ij}$$

and has zero stress at the surface of the wire *up to terms of order* k^3 *and* $k\omega^2$, with $\omega = ck$ and $c = \sqrt{Y/\rho}$.

Thus longitudinal sound down a thin wire travels with a speed of sound set by Young's modulus.

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 10, due Monday Nov. 17
Latest revision: December 18, 2003, 3:14 pm

Reading

Feynman, I.1 Atoms in Motion, I.6 Probability, & I.43 Diffusion

Schroeder, chapter 1. Browse section 6.4 (Maxwell distribution of velocities).

Problems

Schroeder,

(1.12) *How Dilute is Air?* Assume small molecules are around 0.4 nm in diameter. Remember $PV = Nk_BT$ with $k_B = 1.381 \times 10^{-23}$, $P_{atm} \sim 1.013 \times 10^5$Pa, and $T \sim 300K$.

(1.18) *Molecular Velocities.* Remember the kinetic energy $(1/2)m\mathbf{v}^2$ of molecules equals $(3/2)k_BT$. The mass of a N_2 molecule is $m = 2m_{Ni} \sim 28m_p \sim (28gm/N_A) \times 1kg/1000gm = 4.65 \times 10^{-26}kg$.

(1.33) *P-V diagram.* Make a table with rows A, B, C, and "Whole Cycle" and columns "Work done on gas", "Change in Energy content of gas", and "Heat added to gas". Relevant formulas: Work on gas = -$\int P\,dV$; energy in gas $U = Nf(k_BT/2) = (Nf/2)PV$.

(1.60) *Frying Pan.* Do this three ways. (a) Guess the answer from your own experience. If you've always used aluminum pans, consult a friend.

(b) Use an argument analogous to Schroeder's equation (1.71) to get a rough answer. Note: For iron, the specific heat $c_p = 450J/kg \cdot C$, the density $\rho = 7900kg/m^3$, and the thermal conductivity $k_t = 80W/m{\cdot}C$. You need to transport heat $c_p\rho V\Delta T$ across an area $A = V/\Delta x$. How much heat will flow across that area per unit time, if the temperature gradient is roughly assumed to be $\Delta T/\Delta x$? How long δt will it take to transport the amount needed to heat up the whole handle?

(c) Roughly model the problem as the time needed for a pulse of heat at $x = 0$ on an infinite rod to spread out a distance equal to the length of the handle, and use the Greens function for the heat diffusion equation (problems 10.3 and 10.4 below). How long until the pulse spreads out a root-mean square distance $\sigma(t)$ equal to the length of the handle?

(10.1) Random walks in Grade Space. Let's make a simple model of the prelim grade distribution. Let's imagine a multiple-choice test of ten problems of ten points each. Each problem is identically difficult, and the mean is 70. How much of the point spread on the exam is just luck, and how much reflects the differences in skill and knowledge of the people taking the exam? To test this, let's imagine that all students are identical, and that each question is answered at random with a probability 0.7 of getting it right. What is the expected mean and standard deviation for the exam? (Work it out for one question, and then use our theorems for a random walk with ten steps.) A typical exam with a mean of 70 might have a standard deviation of about 15. What physical interpretation do you make of the ratio of the random standard deviation and the observed one?

(10.2) Probability Distributions. I'm assuming you're familiar with probabilities for discrete events (like coin flips and card games), but you probably haven't worked much with probability distributions for continuous variables (like human heights and atomic velocities). The three probability distributions most commonly encountered in physics are: (*i*) **Uniform:** $\rho_{\text{uniform}}(x) = 1$ for $0 \leq x < 1$, $\rho(x) = 0$ otherwise; produced by random number generators on computers. (*ii*) **Exponential:** $\rho_{\text{exponential}}(t) = e^{-t/\tau}/\tau$ for $t \geq 0$, familiar from radioactive decay and used in the collision theory of gases. (*iii*) **Gaussian:** $\rho_{\text{gaussian}}(v) = e^{-v^2/2\sigma^2}/(\sqrt{2\pi}\sigma)$, describing the probability distribution of velocities in a gas, the distribution of positions at long times in random walks, the sums of random variables, and the solution to the diffusion equation.

(a) **Likelihoods.** What is the probability that a random number uniform on $[0, 1)$ will happen to lie between $x = 0.7$ and $x = 0.75$? That the waiting time for a radioactive decay of a nucleus will be more than twice the exponential decay time τ? That your score on an exam with Gaussian distribution of scores will be greater than 2σ above the mean? (Note: $\int_2^\infty (1/\sqrt{2\pi}) \exp(-v^2/2)\, dv = (1 - \text{erf}(\sqrt{2}))/2 \sim 0.023$.)

(b) **Normalization, Mean, and Standard Deviation.** Show that these probability distributions are normalized: $\int \rho(x)dx = 1$. What is the mean x_0 of each distribution? The standard deviation $\sqrt{\int (x - x_0)^2 \rho(x) dx}$? (Hint: $\int_{-\infty}^\infty (1/\sqrt{2\pi}) \exp(-x^2/2)\, dx = \int_{-\infty}^\infty x^2 (1/\sqrt{2\pi}) \exp(-x^2/2)\, dx = 1$).

(c) **Sums of variables.** Draw a graph of the probability distribution of the sum $x + y$ of two random variables drawn from a uniform distribution on $[0, 1)$. Argue in general that the sum $z = x + y$ of random variables with distributions $\rho_1(x)$ and $\rho_2(y)$ will have a distribution given by the *convolution* $\rho(z) = \int \rho_1(x)\rho_2(z - x)\, dx$.

(d) **Multidimensional probability distributions.** In statistical mechanics, we often discuss probability distributions for many variables at once (for example, all the components of all the velocities of all the atoms in a box). Let's consider just the probability distribution of one molecule's velocities. If v_x, v_y, and v_z of a molecule are all distributed with a Gaussian distribution with $\sigma = \sqrt{kT/M}$ (Feynman's equation 40.9, next week), then we describe the combined probability distribution as a function of three variables as the product of the three Gaussians:

$$\begin{aligned}\rho(v_x, v_y, v_z) &= 1/(2\pi(kT/M))^{3/2} \exp(-m\mathbf{v}^2/2kT) \\ &= \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_x^2}{2kT}}\right) \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_y^2}{2kT}}\right) \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_z^2}{2kT}}\right).\end{aligned}$$

Show, using your answer for the standard deviation of the Gaussian in part (b), that the mean kinetic energy is $kT/2$ per dimension. Show that the probability that the speed is $v = |\mathbf{v}|$ is given by a Maxwellian distribution

$$\rho_{\text{Maxwell}}(v) = \sqrt{2/\pi}(v^2/\sigma^3)\exp(-v^2/2\sigma^2).$$

(Hint: What is the probability that $|\mathbf{v}|$ is between v_r and $v_r + \Delta r$, for small Δr? The area of a sphere of radius R is $4\pi R^2$.)

(e) Assuming the probability distribution for the z component of velocity given in part (d), $\rho(v_z) = \left(\sqrt{\frac{M}{2\pi kT}} e^{\frac{-Mv_z^2}{2kT}}\right)$, give the probability density that an N_2 molecule will have a vertical component of the velocity equal to the escape velocity from the Earth (about 10 km/sec, if I remember right). Do we need to worry about losing our atmosphere? (Hint: this is closely related to Schroeder's problem 1.18.) Optional: Try the same calculation for H_2, where you'll find a substantial leakage. You'll want to know that there are 3×10^{16} seconds in a billion years, and molecules collide (and scramble their velocities) many times per second. That's why Jupiter has hydrogen gas in its atmosphere, and Earth does not.

(10.3) Thermal Diffusion. The rate of energy flow in a material with thermal conductivity k_t and a temperature field $T(x, y, z, t) = T(\mathbf{r}, t)$ is $\mathbf{J} = -k_t \nabla T$ (see Feynman eq. 43.41). Energy is locally conserved, so the energy density E satisfies $\partial E/\partial t = -\nabla \cdot \mathbf{J}$.

(a) If the material has constant specific heat c_p and density ρ, so $E = c_p \rho T$, show that the temperature T satisfies the diffusion equation $\partial T/\partial t = \frac{k_t}{c_p \rho}\nabla^2 T$. (See Schroeder, problem 1.62).

(b) By putting our material in a cavity with microwave standing waves, we heat it with a periodic modulation $T = \sin(kx)$ at $t = 0$, at which time the microwaves are turned off. Show that amplitude of the temperature modulation decays exponentially in time. How does the amplitude decay rate depend on wavelength $\lambda = 2\pi/k$?

(10.4) Heat Diffusion Spot. The diffusion equation for the heat density in a two-dimensional sheet is

$$\partial q/\partial t = K(\partial^2 q/\partial x^2 + \partial^2 q/\partial y^2).$$

(a) **Diffusion in Two Dimensions.** Show that if $f(x, t)$ satisfies the diffusion equation in one dimension, then $f(x, t)f(y, t)$ solves the diffusion equation in two dimensions. (Related formulæ: Product Rule, $\partial fg/\partial z = \partial f/\partial z\, g + f\, \partial g/\partial z$.)

(b) **The heat spot.** A screen of thermal diffusion constant K is heated at $x = y = 0$ and $t = 0$ by a thin laser beam pulse. The total heat deposited is Q. Use part (A) and the Greens function for the one dimensional diffusion equation to derive the equation for $q(x, y, t)$, the heat density after a time t. What is the root-mean-square radius $r(t) = \sqrt{\langle x^2 + y^2 \rangle}$ for this spot? (Related formulæ: $\partial\rho/\partial t = D\partial^2\rho/\partial x^2$; If $\rho(x, 0) = \delta(x)$, $\rho(x, t) = G(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$ and $\langle x^2 \rangle = 2Dt$; $\langle f(\mathbf{z})\rangle = \int f(\mathbf{z})\rho(\mathbf{z})\, d^D z$.)

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 11, due Monday Nov. 24
Latest revision: December 18, 2003, 3:14 pm

Reading

Feynman, I.39 The Kinetic Theory of Gases, I.40 Principles of Statistical Mechanics, I.41 The Brownian Movement, & I.42 Applications of Kinetic Theory.

Schroeder, "Very Large Numbers" subsection of section 2.4, 2.5 (Ideal Gas) & 3.1 (Temperature).

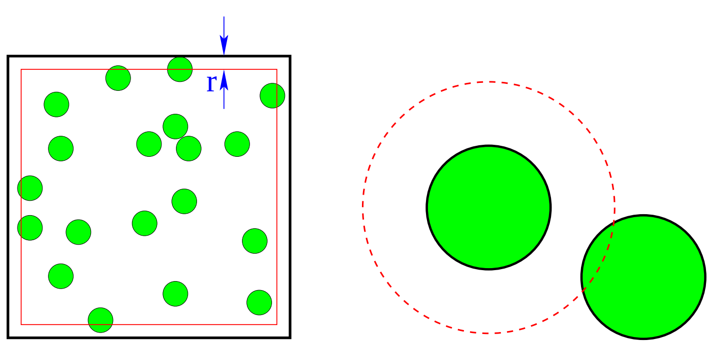
Web reading

Introduction to the Cosmic Microwave Background Radiation:
http://background.uchicago.edu/∼whu/beginners/introduction.html
http://background.uchicago.edu/∼whu/intermediate/intermediate.html,
especially the parts Acoustic Oscillations, Angular Peaks, and First Peak.

Problems

Schroeder,
(3.2) *Zeroth law.*
(3.3) *Entropy graphs.*

(11.1) Entropy and Hard Spheres.



We can improve on the realism of the ideal gas by giving the atoms a small radius. If we make the potential energy infinite inside this radius ("hard spheres"), the potential energy is simple (zero unless the spheres overlap, which is forbidden). Let's do this in two dimensions.

A two dimensional $L \times L$ box contains an ideal gas of N hard disks of radius $r \ll L$ (left figure). The disks are dilute: the summed area $N\pi r^2 \ll L^2$. Since the disks cannot be within r of the edges of the box, let A be the effective volume allowed for the first disk in the box: $A = (L - 2r)^2$.

(a) **Configuration Space Volume for Hard Disks.** The area allowed for the second disk is $A-\pi(2r)^2$ (right figure), ignoring the small correction when the excluded region around the first disk overlaps the excluded region near the walls of the box. The area allowed for the n^{th} disk is $A-(n-1)\pi(2r)^2$, ignoring corrections for the overlaps of the excluded regions. Let configuration space $\mathbf{X}$ be the $2N$ dimensional space of positions $\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\ldots \mathbf{x}^{(N)}$. Write an expression for the volume $\Omega_{\mathbf{X}}$ of allowed zero-energy configurations of hard disks, in the configuration space $\mathbf{X}$, ignoring the overlapping excluded regions. (*Related formulæ*: For a 3D ideal gas, $\Omega_{\mathbf{P}} = (\pi' s)(2mE)^{(3N-1)/2}$, $\Omega_{\mathbf{X}} = V^N/N!$.)

(b) **Statistical Mechanical Entropy for Hard Disks.** It's now easy to write the configurational entropy, $S_{\mathbf{X}}$ for the hard disks of part (a) as a sum over n. Use the "Math truth" below to find a formula for the entropy that does not involve a sum over n, accurate to first order in the area of the disks πr^2. (*Related formulæ*: $S = k_B \ln(\Omega)$; **Simpson's Rule:** $n! \approx (n/e)^n \sqrt{2\pi n}$; **Math Truth**: To first order in ϵ, $\sum_{n=1}^{N} \log\left(A-(n-1)\epsilon\right) = N \log\left(A-(N-1)\epsilon/2\right)$.)

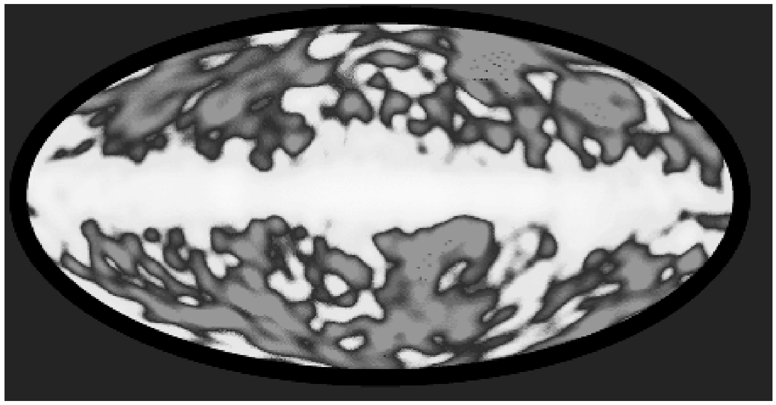
(c) **Pressure for Hard Disks.** Assume the hard-disk configurational entropy is $S_{\mathbf{X}} = N k_B \log(A - Nb)$ for some area b, representing the effective excluded area due to the other disks. (Your answer to (b) won't quite have this form, but it's a good approximation, up to an overall N-dependent constant.) Just as for the ideal gas, the internal energy U is purely kinetic, and the kinetic energy and momentum-space entropy depend only on temperature and not on volume. So, if we isothermally expand this hard-disk gas from initial area A_1 to A_2, the internal energy doesn't change: $\Delta U = Q + W = 0$, so the heat Q added to the gas equals $-W$, the work done by the gas expanding against the external pressure P. By differentiating with respect to A_2, find the pressure for the hard-sphere gas. (Hint: for $b = 0$ it should reduce to the ideal gas law.) (*Related formulæ*: $W = -\int_{A_1}^{A_2} P dA$ and $\Delta S = Q/T$ (Thermodynamic Entropy). For a 3D ideal gas, $PV = N k_B T$ and $U = 3/2\, N k_B T$.)

(11.2) Waves and the Birth of the Universe. (Large thanks to Ira Wasserman: errors are of course my own.)
Our universe started very hot and dense, in what we call the Big Bang. This auspicious starting point is what sets our arrow of time.

Because the universe is expanding, the light emitted back then has redshifted (due to the Doppler effect), so the immensely hot and bright origin of the universe now resides in a microwave background radiation that you'd get from a black body at a temperature of 3 K. We've learned a lot about our universe recently by carefully measuring the differences between the temperatures of this radiation as we look in different directions in the night sky.

Figure (11.2.1) shows these tiny fluctuations in temperature.* The fluctuations in temperature represent noisy thermal waves in the early universe.

* Actually, it shows these fluctuations after a dipole term has been subtracted out.



(11.2.1) Microwave background radiation map. Variation in temperature of the microwave background radiation, after the constant term and the dipole term are subtracted out, from COBE, the Cosmic Microwave Background Explorer. The fluctuations are about one part in 100,000. The bright stripe at the equator is our galaxy.

Because it was still very hot, all the hydrogen in the universe was still ionized. Light doesn't travel very far in ionized gases (it accellerates the charges and scatters from them): the light and matter remained in equilibrium with one another until the universe was around 300,000 years old, when it got cold enough for the electrons and protons to combine into hydrogen.

Before 300,000 years, the combined light-and-matter density satisfied a wave equation:

$$(1 + R)\partial^2\Theta/\partial t^2 = (c^2/3)\nabla^2\Theta, \tag{1}$$

where c is the speed of light in vacuum, Θ is the temperature fluctuation $\Delta T/T$, t is "conformal" time (treat it as regular time), and R is the contribution of matter to the density. Θ can be viewed also as the energy density fluctuations $\Delta e/e$ where $e = U/V$ is the energy density: denser regions are hotter. After recombination, the light was able to travel directly (albeit red-shifted) to our cameras. So, the microwave background radiation is giving us a snapshot of the temperature fluctuations of the universe at age 300,000 years.

(a) What is the speed of sound in this gas?

Let's derive equation (1).

(b) The dominant contribution to the pressure of this combined light-and-matter mixture is due to the light pressure. Feynman (section 39.3) shows that the photon gases satisfy $PV^{4/3} = C$, where C is some constant. (Photons are quantized light particles: you'll learn about them more in Modern Physics.) The bulk modulus B is defined by

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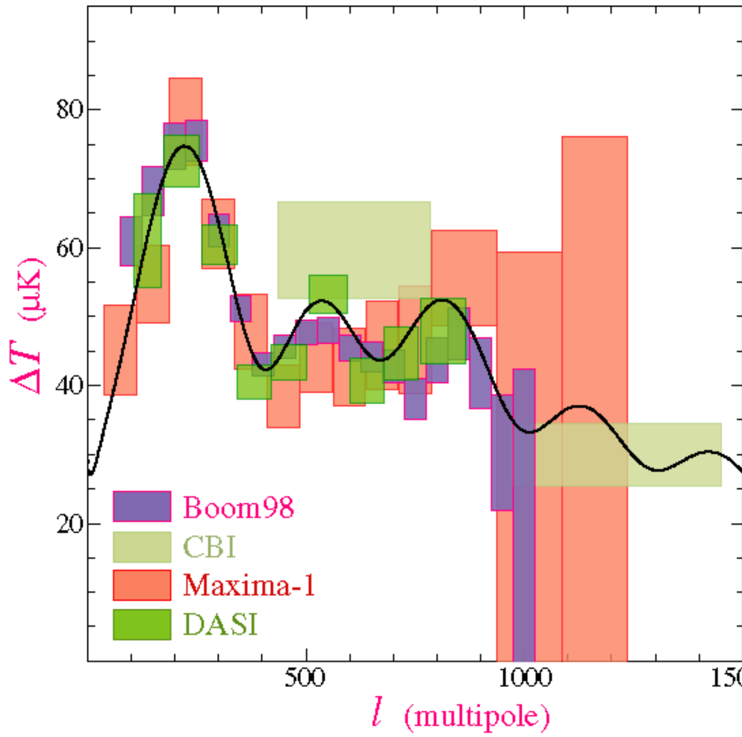
The dipole comes from the Doppler effect of *our* motion. Einstein's theory states that all motion is relative: the laws of physics don't depend upon how fast the Sun is moving with respect to the distant galaxies. But that doesn't mean that the distant galaxies (or, even better, the glow from the Big Bang) doesn't have a particular velocity! We can measure our velocity with respect to the universe by using this dipole.



(11.2.2) Wave vector dependence of microwave radiation pattern. Variation in temperature of the microwave background radiation, decomposed into spherical harmonics. Spherical harmonics are like a Fourier transform, but for angles: you can think of ℓ for the multipole as roughly corresponding to wavenumber k of the corresponding temperature fluctuation in the universe when it became transparent to photons (at recombination) (From Wayne Hu's Web site, above).

$\Delta P = -B\Delta V/V$. Show that the bulk modulus is $4P/3$. Feynman also shows that $PV = U/3$. Let P_0 be the average light pressure, U_0 be the average energy in the light in an initial volume V_0. Show (trivially) that $B = (4/9)U_0/V_0$ where U_0/V_0 is the average photon energy density.

(c) The total mass density for the wave equation ρ in the early universe has three important contributions. First, there is the regular mass of particles (mostly baryons) M_{baryon}/V_0. Then there is the energy density of the photons divided by c^2 (remember $E = mc^2$?), $U_0/(V_0c^2)$. Finally, there is a contribution due to the pressure P_0/c^2 (this is really a component of a stress-energy tensor...) Show that the total density is $\rho = M_{\text{baryon}}/V_0 + 4U_0/(3V_0c^2)$.

(d) Derive equation (1) above from $\rho\partial^2 P/\partial t^2 = B\nabla^2 P$. What is the formula for R?

A theory called "inflation" predicts that at very early times the universe was left in a state which we can think of as being uniform in temperature and density, but with a random velocity field. Let's derive what the density field $\Theta(\mathbf{x})$ should look like at time $t = 300,000$ years.

(e) Consider first an initial standing-wave perturbation $\Theta(\mathbf{x}, t) = \tilde{\Theta}_k \sin(\mathbf{k}\cdot\mathbf{x}) \sin(\omega_k t)$. (Of course the universe started with a superposition of many such standing waves with different **k**.) What is ω_k? The energy density fluctuation is zero at $t = 0$, as inflation predicts. At what times will this wave have maximum fluctuations? Which values of $k = |\mathbf{k}|$ will have largest fluctuations at $t = 300,000$ years? Show that odd multiples of the first peak are maxima, while even multiples are minima. Assuming for simplicity that $R = 0$ (photon-dominated mass density), give the wavelength of the first peak, in light years.

Our picture of the background radiation (first above) is a cross section of the original radiation at a sphere given by the 10 billion years since recombination (modulo corrections due to the age of the universe). Since the data is on a sphere, they need to decompose our data into spherical harmonics: the constant ℓ in the wave-vector figure (II.2.2) roughly corresponds to wave number k.

(f) Is twice the ℓ value of the first maximum in figure (11.2.2) a maximum or a minimum? Does that agree with your conclusion for part (e)? What about three times the first maximum? (The full theory includes other effects which shift the peak positions.)

Physics 218: Waves and Thermodynamics
Fall 2003, James P. Sethna
Homework 12, due Wednesday Dec. 3
Latest revision: December 18, 2003, 3:14 pm

Reading

Feynman, I.44 Laws of Thermodynamics, I.45 Illustrations of Thermodynamics, & I.46 Ratchet and Pawl

Schroeder, 2.6 (Entropy), 3.1 (Temperature), & 4.1 (Heat Engines. Browse the rest of chapter 4 (Engines and Refrigerators).

For further reading (much more advanced)

Freeman J. Dyson, "Time without end: Physics and biology in an open universe", *Reviews of Modern Physics* **51**, 447 (1979). (Download from prola.aps.org).

Problems

Schroeder,

(3.10) *Entropy and Ice Cubes.* The latent heat of ice is 80 cal/g, and the specific heat of water is $c_p = 1cal/(gm \cdot K)$; one calorie is 4.186 J.

(3.16) *Entropy and bits.*

(12.1) Life and the Heat Death of the Universe.

Freeman Dyson discusses how living things might evolve to cope with the cooling and dimming we expect during the heat death of the universe.

Dyson models an intelligent being as a heat engine that generates a fixed entropy ΔS per thought. (This correspondence of information with entropy is a standard idea from computer science.)

(a) **Energy needed per thought.** Assume that the being draws heat Q from a hot reservoir at T_1 and radiates it away to a cold reservoir at T_2. What is the minimum energy Q needed per thought, in terms of ΔS and T_2? You may take T_1 very large. (*Related formulæ*: For Carnot engine, $\Delta S = Q_2/T_2 - Q_1/T_1 = 0$; First Law: $Q_1 - Q_2 = W$ (energy is conserved).)

(b) **Time needed per thought to radiate energy.** Dyson shows, using theory not important here, that the power radiated by our intelligent–being–as–heat–engine is no larger than CT_2^3, a constant times the cube of the cold temperature.* Write an expression for the maximum rate of thoughts per unit time dH/dt (the inverse of the time Δt per thought), in terms of ΔS, C, and T_2.

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* The constant scales with the number of electrons in the being, so we can think of our answer Δt as the time per thought per mole of electrons.

(c) **Number of thoughts for an ecologically efficient being.** Our universe is expanding: the radius R grows roughly linearly in time t. The microwave background radiation has a characteristic temperature $\Theta(t) \sim R^{-1}$ which is getting lower as the universe expands: this red-shift is due to the Doppler effect. An ecologically efficient being would naturally try to use as little heat as possible, and so wants to choose T_2 as small as possible. It cannot radiate heat at a temperature below $T_2 = \Theta(t) = A/t$. How many thoughts H can an ecologically efficient being have between now and time infinity, in terms of ΔS, C, A, and the current time t_0?

(d) **Time without end: Greedy beings.** Dyson would like his beings to be able to think an infinite number of thoughts before the universe ends, but consume a finite amount of energy. He proposes that his beings need to be profligate in order to get their thoughts in before the world ends: he proposes that they radiate at a temperature $T_2(t) \sim t^{-3/8}$ which falls with time, but not as fast as $\Theta(t) \sim t^{-1}$. Show that with Dyson's cooling schedule, the total number of thoughts H is infinite, but the total energy consumed U is finite.

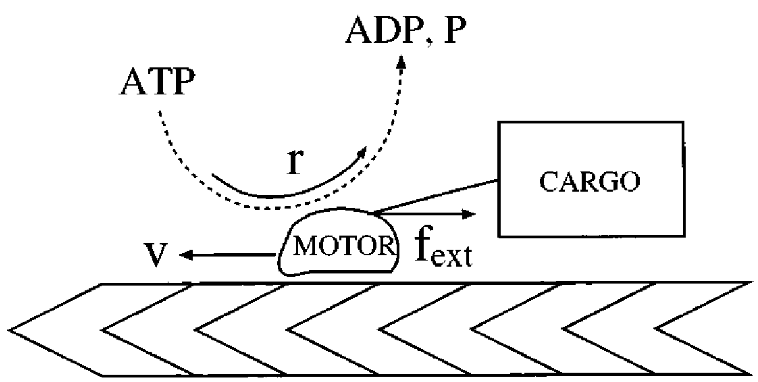

Figure (12.2.1) Cartoon of a motor protein, from Jülicher, Ajdari, and Prost, *Rev. Mod. Phys.* **69**, 1269 (1997). As it carries some cargo along the way (or builds an RNA or protein, ...) it moves against an external force f_{ext} and consumes r ATP molecules, which are hydrolized to ADP and phosphate (P).

(12.2) Ratchet and Molecular Motors.
Feynman's ratchet and pawl discussion obviously isn't so relevant to machines you can make in your basement shop. The thermal fluctuations which turn the wheel to lift the flea are too small to be noticable on human length and time scales (you need to look in a microscope to see Brownian motion). On the other hand, his discussion turns out to be surprisingly close to how real cells move things around. Physics professor Michelle Wang studies these molecular motors in the basement of Clark Hall.

Inside your cells, there are several different molecular motors, which move and pull and copy (figure 12.2.1). There are molecular motors which contract your muscles, there are motors which copy your DNA into RNA and copy your RNA into protein, there are motors which transport biomolecules around in the cell. All of these motors share some common

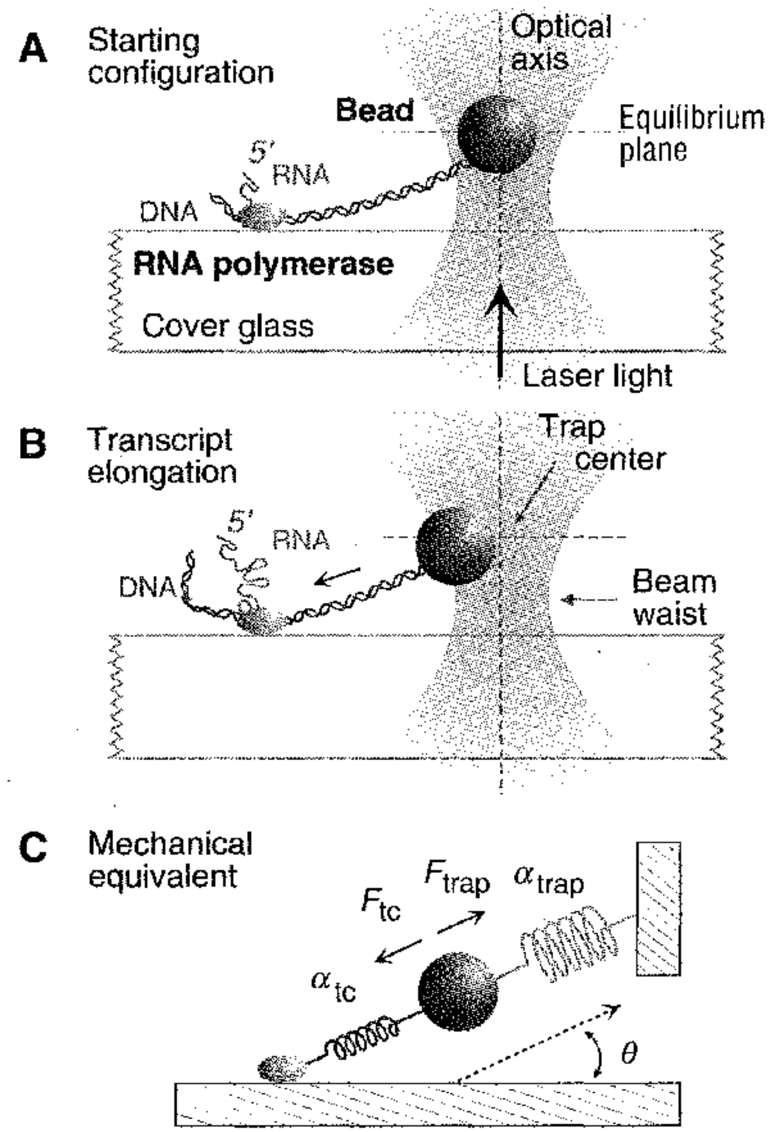

Figure (12.2.2) Cartoon of Professor Wang's early laser tweezer experiment, (Yin, Wang, Svoboda, Landick, Block, and Gelles, *Science* **270**, 1653 (1995)). (A) The laser beam is focused at a point (the "laser trap"); the polystyrene bead is pulled (from dielectric effects) into the intense part of the light beam. The "track" is a DNA molecule attached to the bead, the motor is an RNA polymerase molecule, the "cargo" is the glass cover slip to which the motor is attached. (B) As the motor (RNA polymerase) copies DNA onto RNA, it pulls the DNA "track" toward itself, dragging the bead out of the trap, generating a force resisting the motion. (C) A mechanical equivalent, showing the laser trap as a spring and the DNA (which can stretch) as a second spring.

features: (1) they move along some linear track (microtubule, DNA, ...), hopping forward in discrete jumps between low-energy positions, (2) they consume energy (burning ATP or NTP) as they move, generating an effective force pushing them forward, and (3) their mechanical properties can be studied by seeing how their motion changes as the external

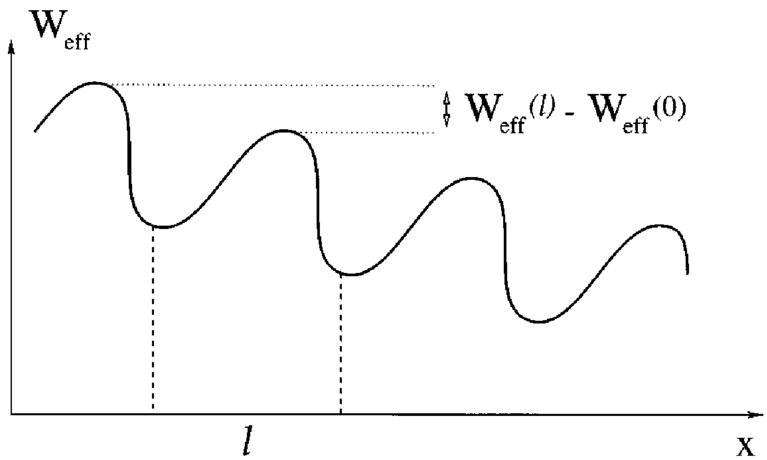

Figure (12.2.3) The effective potential for moving along the DNA (from Prost, above). Ignoring the tilt W_e, Feynman's energy barrier ϵ is the difference between the bottom of the wells and the top of the barriers. The experiment changes the tilt by adding an external force pulling ℓ to the left. In the absence of the external force, W_e is the (Gibbs free) energy released when one NTP is burned and one RNA nucleotide is attached.

force on them is changed (figure 12.2.2).

For transcription of DNA into RNA, the motor moves on average one base pair (A, T, G or C) per step: $\Delta\ell$ is about 0.34nm. We can think of the triangular grooves in the ratchet as being the low-energy states of the motor when it is resting between steps. The barrier between steps has an asymmetric shape (figure 12.2.3), just like the energy stored in the pawl is ramped going up and steep going down. Professor Wang showed (in a later paper) that the motor stalls at an external force of about 27 pN (pico-Newton).

(a) At that force, what is the energy difference between neighboring wells due to the external force from the bead? (This corresponds to $L\theta$ in Feynman's ratchet.) Let's assume that this force is what's needed to balance the natural force downhill that the motor develops to propel the transcription process. What does this imply about the ratio of the forward rate to the backward rate, in the absence of the external force from the laser tweezers, at a temperature of $300K$, (from Feynman's discussion preceding equation 46.1)? ($k_B = 1.381 \times 10^{-23}$J/K).

The natural force downhill is coming from the chemical reactions which accompany the motor moving one base pair: the motor burns up an NTP molecule into a PP_i molecule, and attaches a nucleotide onto the RNA. The net energy from this reaction depends on details, but varies between about 2 and 5 times 10^{-20} Joule. This is actually a Gibbs free energy difference, but for this problem treat it as just an energy difference.

(b) The motor isn't perfectly efficient: not all the chemical energy is available as motor force. From your answer to part (a), give the efficiency of the motor as the ratio of force-times-distance produced to energy consumed, for the range of consumed energies given.

(12.3) Carnot Refrigerator. Our refrigerator is about $2m \times 1m \times 1m$, and has insulation about $3cm$ thick. The insulation is probably polyurethane, which has a thermal conductivity of about 0.02 W/(m K). Assume that the refrigerator interior is at 270K, and the room is at 300K.

(a) How many watts of energy leak from our refrigerator through this insulation?

Our refrigerator runs at 120 V, and draws a maximum of 4.75 amps. The compressor motor turns on every once in a while for a few minutes.

(b) Suppose (*i*) we don't open the refrigerator door, (*ii*) the thermal losses are dominated by the leakage through the foam and not through the seals around the doors, and (*iii*) the refrigerator runs as a perfectly efficient Carnot cycle. How much power on average will our refrigerator need to operate? What fraction of the time will the motor run?

(12.4) Entropy of Glasses. Glasses aren't really in equilibrium. In particular, they do not obey the third law that the entropy S goes to zero as the temperature approaches absolute zero. Experimentalists measure a "residual entropy" by subtracting the entropy change from the known entropy $S_{\text{equilibrium}}(T)$ at high temperatures (say, in the ordinary equilibrium liquid state):

$$S_{\text{residual}} = S_{\text{equilibrium}}(T) - \int_0^T \frac{dQ}{T\,dT} dT.$$

Usually, one calls dQ/dT the specific heat C of the material, but we're being fussy:

(a) If you put a glass in an insulated box, it will warm up (very slowly) because of microscopic atomic rearrangements which lower the potential energy. So, glasses don't have a well-defined temperature or specific heat. In particular, the heat flow upon cooling and on heating $\frac{dQ}{dT}(T)$ won't precisely match (although their integrals will agree by conservation of energy). By using the second law (entropy can only increase), show that the residual entropy measured on cooling is always less than the residual entropy measured on heating.*

(b) The residual entropy of a glass is about k_B per molecular unit. It's a measure of how many different glassy configurations of atoms the material can freeze into (section I.46-4). In a molecular dynamics simulation with one hundred indistinguishable atoms, and assuming that the residual entropy is $k_B \log 2$ per atom, what is the probability that two coolings to zero energy will arrive at equivalent atomic configurations (up to permutations)? In a system with 10^{23} molecular units, with residual entropy $k_B \log 2$ per unit, about how many coolings would be needed to arrive at the same configuration twice?

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* See Steve Langer's paper, *Phys. Rev. Lett.* **61**, 570 (1988), although M. Goldstein noticed it earlier.

Physics 218—Exam I (October 3) **Fall 2003**
James P. Sethna

NAME:______________________________

The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

Multiple Choice (40 pts):_____ **x 5pts =** _______________

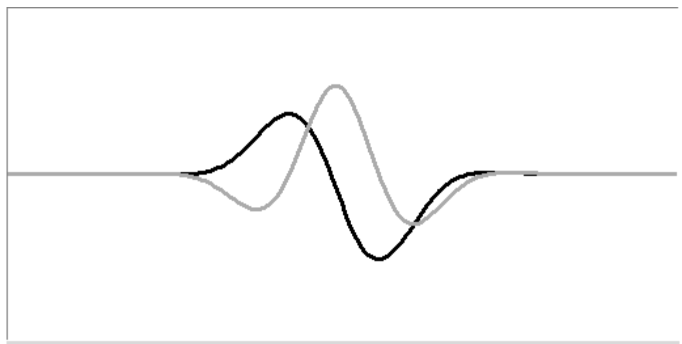
Short Answer: **S1 (30 pts)** _______________

S2 (30 pts) _______________

TOTAL __________________

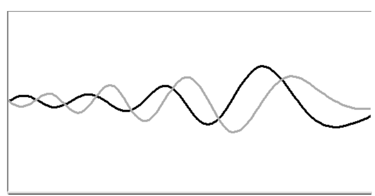
Multiple Choice: Be sure to put answers in boxes provided. (Sorry: no partial credit!)

M1. (25 pts) Waves on Strings: Heights and and Velocities.



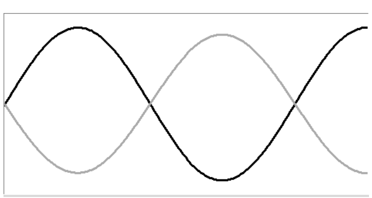
The figure above shows a packet travelling to the right on a horizontal stretched string. The black line shows the height $\eta(x,t)$, and the grey line shows the velocity $\frac{\partial \eta}{\partial t}(x,t)$ Of the six figures below, five represent the shape of this pulse at a later time, under various circumstances. In the blanks after the five circumstances (A-E) below, give the number of the corresponding figure (1-6). (5 points each)

(1)

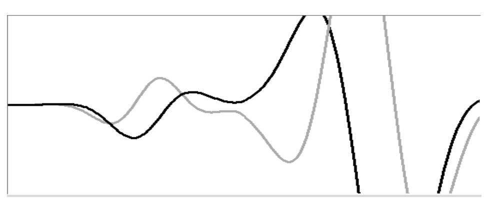


(2)

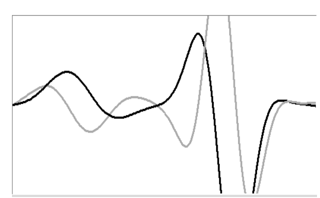
(3)



(4)



(5)

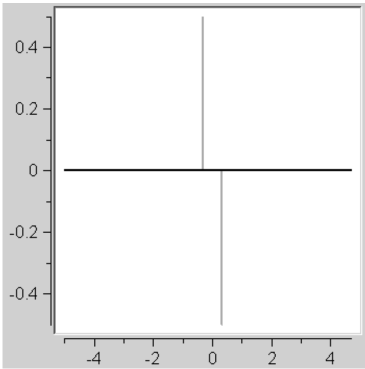


(6)

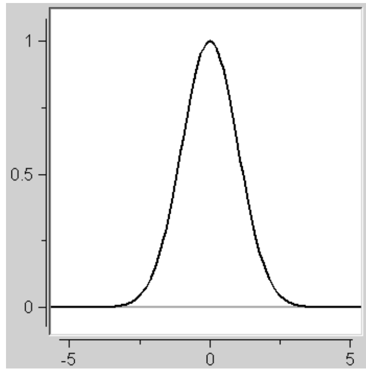
(A) Middle of the packet hitting fixed boundary on right. _____
(B) Middle of the packet hitting free boundary on right. _____
(C) Chirping after a long propagation time due to dispersion. _____
(D) Reflection and transmission at step up in mass density of string (at center). _____
(E) Reflection and transmission at step down in mass density of string (at center). _____

Warning: Incident pulse is roughly *antisymmetric*, so it acts differently than you may remember for symmetric pulses.

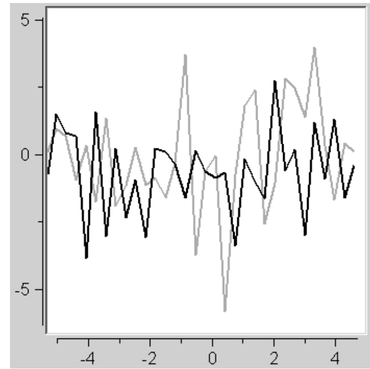
M2. (15 pts) Fourier.



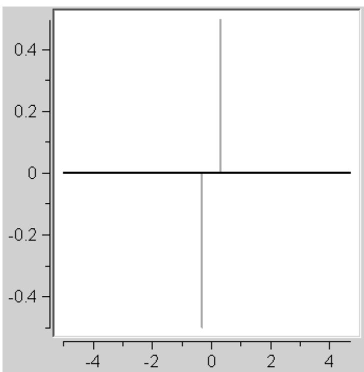
(1)



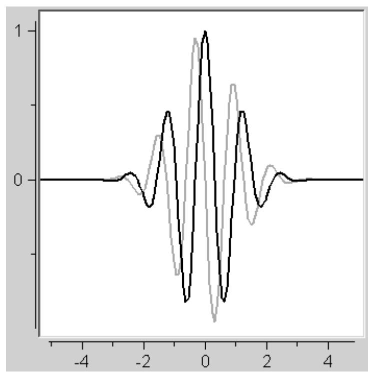
(2)



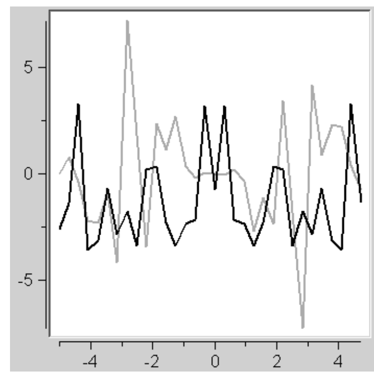
(3)



(4)

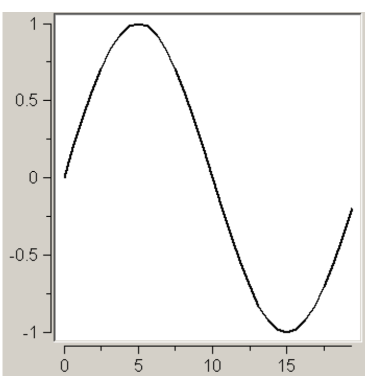


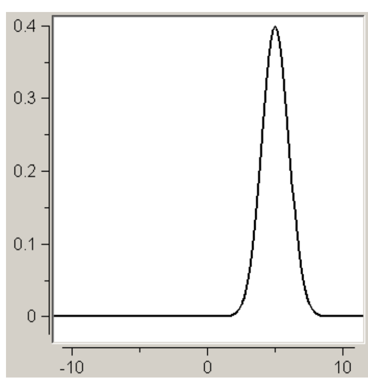
(5)

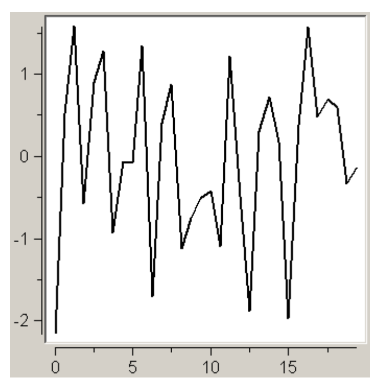


(6)

Which figure (1-6) above is the Fourier representation of the following three figures below? (Dark line is real part, lighter line is imaginary part.)







(A) _____ (B) _____ (C) _____

Short Answer: Show Your Work

S1. (30 pts) Energy Conservation: Reflection and Transmission.

(A) Pulses. An incident pulse $\eta_I(x,t) = f(x/c_1 - t)$ is moving right on a horizontal stretched string with tension τ towards the origin at $x = 0$. At $x = 0$, there is a (massless) discontinuity in the mass density of the string, such that the sound velocity changes from c_1 for $x < 0$ to c_2 for $x > 0$. Assume the reflection and transmission coefficients are R and T for the pulse.

Write a formula for the reflected pulse $\eta_R(x,t)$ and transmitted pulse $\eta_T(x,t)$, leaving your result in terms of f, R, T, c_1, and c_2. For your and our convenience, do not solve for R and T in terms of the two speeds of sound. Also, write the equation for R involving T that guarantees that the string is continuous at $x = 0$.*

$\eta_R(x,t) =$____________

$\eta_T(x,t) =$____________

$R =$____________ is implied by continuity of string.

* Hint: To check your answer for R, see if $R = (c_2 - c_1)/(c_2 + c_1)$ and $T = 2c_2/(c_2 + c_1)$ satisfies your equation.

(B) Energy. Write expressions for the total energy (the integral of the energy density E) for the incident pulse $\mathcal{E}_I$, the transmitted pulse $\mathcal{E}_T$, and the reflected pulse $\mathcal{E}_R$. Change variables and simplify until each becomes a constant times $\int f'(z)^2\,dz$. Your constants may involve R, T, τ, c_1, and c_2, but should not need to involve any other parameters.

$\mathcal{E}_{\mathcal{I}} =$ ____________ $\int (f'(z))^2 \; dz$

$\mathcal{E}_{\mathcal{R}} =$ ____________ $\int (f'(z))^2 \; dz$

$\mathcal{E}_{\mathcal{T}} =$ ____________ $\int (f'(z))^2 \; dz$

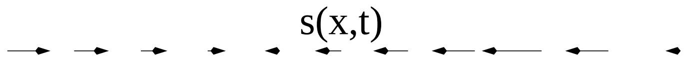
(C) Conservation of Energy. The discontinuity in the string neither holds nor dissipates energy, so the energy of the incident pulse must equal the sum of the energy of the transmitted and reflected pulses. Using your solution to part (B), give an equation for R^2 involving c_1, c_2, and T that expresses this conservation of energy.* (Please don't plug in your last formula from part (A): we want the condition just for energy conservation, not including continuity of string.)

$R^2 =$____________ is implied by energy conservation.

---

* Hint: To check your answer, see if $R = (c_2 - c_1)/(c_2 + c_1)$ and $T = 2c_2/(c_2 + c_1)$ satisfies your equation.

S2. (30 pts) Deriving the Wave Equation for Sound in Air.



In deriving the wave equation for sound, we made a linear approximation to the pressure dependence on the volume ($P \approx P_0 - B\Delta V/V$). This is just the first term in a Taylor series.* Let's find out what happens when we add the first nonlinear term,

$$P = P_0 - B_1 \frac{\Delta V}{V} + B_2 \left(\frac{\Delta V}{V} \right)^2 . \qquad (S2.1),$$

where, as before, $\frac{\Delta V}{V} = \partial s/\partial x$ for a longitudinal wave varying only along the $\hat{x}$ axis.

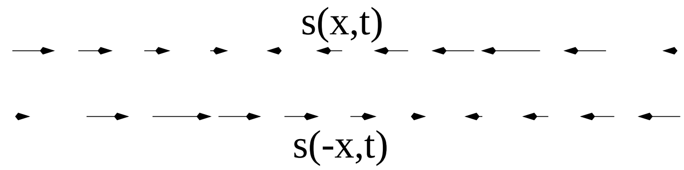
(A) Sound in Air: First Nonlinear Terms. Derive the equation of motion for the longitudinal displacement field $s(x,t)$ in air of density ρ, keeping the first nonlinear correction proportional to B_2 above.

$\frac{\partial^2 s}{\partial t^2} =$______________________.

* For dilute gasses the full dependence has $P \sim V^{-\gamma}$.

(B) Sound in Air: Symmetry Analysis.

The laws for sound in air have five of the six symmetries associated with waves on strings. They have the three continuous symmetries: they are homogeneous (translation invariant, $x \to x + \Delta$), time independent ($t \to t + \Delta$), and are invariant under an overall shift in the displacement field $s \to s + \Delta$.† They have two of the discrete symmetries of waves on strings: they are reflection invariant ($x \to -x$), and time reversal invariant ($t \to -t$). They do not have an independent symmetry for reversing the sign of the displacement field: the transformation properties of s under reflection and time reversal symmetry are intertwined, since s points along the x axis.



Fill in the boxes below, with a $+1$ for those quantities which are unchanged under the symmetry and a -1 for those quantities which change sign under the symmetry. Then check that the term proportional to B_2 in your answer to part (A) above transforms in the same way as the other terms in the wave equation.

Discrete symmetry transformation properties

	s	$\frac{\partial s}{\partial x}$	$\frac{\partial s}{\partial t}$	$\frac{\partial^2 s}{\partial x^2}$	$\frac{\partial^2 s}{\partial t^2}$	$\frac{\partial^2 s}{\partial x \partial t}$	Your (A)
$x \to -x$							
$t \to -t$							

† Check that your answer to part (A) is invariant under these continuous symmetries!

Physics 218—Exam I (October 3) Fall 2003

Formula Sheet

James P. Sethna

Complex Trigonometry. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$. $\sin(A+B) = \sin A\cos B + \cos A\sin B$, $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\exp(iA) + \exp(iB) = \exp(i(A+B)/2)\, 2\cos((A-B)/2)$, $\cos(A) + \cos(B) = 2\cos((A+B)/2)\cos((A-B)/2)$.

Fourier.
$\tilde{y}_m = (1/L)\int_0^L y(x)\exp(-ik_m x)dx$ where $k_m = 2\pi m/L$; $y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m \exp(ik_m x)$.
$\tilde{y}(k) = \int_{-\infty}^{\infty} y(x)\exp(-ikx)\,dx$; $y(x) = (1/2\pi)\int_{-\infty}^{\infty} \tilde{y}(k)\exp(ikx)\,dk$.
$\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-i2\pi m\ell/N)$ with $m = 0, \ldots N-1$; $\tilde{y}_m^{FFT} = \sum_{\ell=0}^{N-1} y_\ell \exp(-ik_m x_\ell)$.
$G(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-(x-x_0)^2/2\sigma^2\right)$; $\tilde{G}(k) = \exp(-ikx_0)\exp(-\sigma^2k^2/2)$.

Orthonormality.
$(1/L)\int_0^L \exp(ik_m x)\exp(ik_n x)\,dx = \delta_{mn}$, where $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$. The Fourier transform of a Gaussian of width σ, $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-x^2/2\sigma^2\right)$ is a Gaussian $\tilde{f}(k) = \exp\left(-\sigma^2k^2/2\right)$ of width $1/\sigma$. The Fourier transform of $f(x-x_0)$ is $\exp(-ikx_0)\tilde{f}(k)$.

Wave Equation $\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$. Traveling wave solution $\eta(x,t) = f(x \pm ct)$. Standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$. Traveling sine wave (Plane wave) $\eta(x,t) = A\sin(kx - \omega t)$, with $\omega/k = c$.

$c = \sqrt{\tau/\lambda_0}$, $K(x,t) = (\lambda_0/2)(\partial\eta/\partial t)^2$, $V(x,t) = (\tau/2)(\partial\eta/\partial x)^2$, $E(x,t) = K(x,t) + V(x,t)$, $P(x,t) = -\tau(\partial\eta/\partial t)(\partial\eta/\partial x)$, $g_x(x,t) = -\lambda_0(\partial\eta/\partial t)(\partial\eta/\partial x)$.

Traveling waves, $\eta(x,t) = f(x \pm ct)$ have special properties: $\partial\eta/\partial x = \pm(1/c)(\partial\eta/\partial t)$, $K(x,t) = V(x,t)$, $E(x,t) = \tau(\partial\eta/\partial x)^2$.

Sound: $c = \sqrt{B/\rho} = 340$ m/s for air. $p = -B(\partial s/\partial x)$, $\partial^2 s/\partial t^2 = -(1/\rho)\partial p/\partial x$.

General Solutions, bounded at infinity.
Finite string: $\eta(x,t) = \sum_{n=1}^{\infty}\sin(k_n x)\left(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\right)$, $\omega_n = ck_n$, $k_n = n\pi/L$. Infinite string: $\eta(x,t) = f(x-ct) + g(x+ct)$; initial displacement $f(x) + g(x)$, initial velocity $c(g'(x) - f'(x))$.

Boundary Conditions. Fixed: $\eta = \partial\eta/\partial t = 0$. Free: $\partial\eta/\partial x = 0$. Both fixed or free: $\omega_n = cn\pi/L$; mixed $\omega_n = c(2n+1)\pi/2L$.

Symmetries. The wave equation is symmetric under (a) Reflection along x, $\tilde{\eta}(x,t) = \eta(-x,t)$, (b) Time reversal, $\tilde{\eta}(x,t) = \eta(x,-t)$, (c) Reflection along y, $\tilde{\eta}(x,t) = -\eta(x,t)$. It is also (d) Homogeneous, $\tilde{\eta}(x,t) = \eta(x-\Delta,t)$, (e) Time independent, $\tilde{\eta}(x,t) = \eta(x,t-\Delta)$, and invariant under (f) Sideways motion, $\tilde{\eta}(x,t) = \eta(x,t) + \Delta$.

Reflection and Transmission (Massless Knot). $R = (Z_1 - Z_2)/(Z_1 + Z_2)$ and $T = 2Z_1/(Z_1+Z_2)$ where impedances $Z_i = \sqrt{\lambda_i\tau_i}$. If the tension is constant, $R = (c_2-c_1)/(c_2+c_1)$ and $T = 2c_2/(c_2+c_1)$

Group and Phase Velocities. Phase velocity $\omega(k)/k$. Group velocity $d\omega/dk$. One dimensional chain $\omega(k) = \sqrt{K/M}\sqrt{2 - 2\cos(ka)}$.

Physics 218—Exam II (Friday November 9) Fall 2003
James P. Sethna

NAME:______________________________

The multiple choice problems have been designed so that, if you are inspired, you can do them quickly without calculations. There are no tricks to speed up the short answer question.

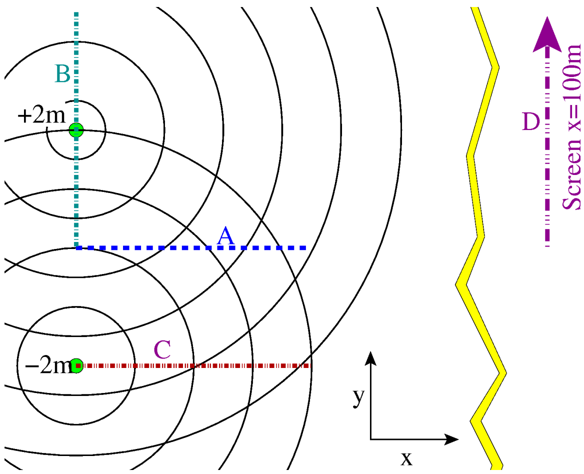
Multiple Choice (50 pts):____ **x 5 pts =** ______________

Short Answer: S1 (50 pts) __________________

TOTAL __________________

Multiple Choice. (Sorry: no partial credit!)

M1. (20 pts) Interference.



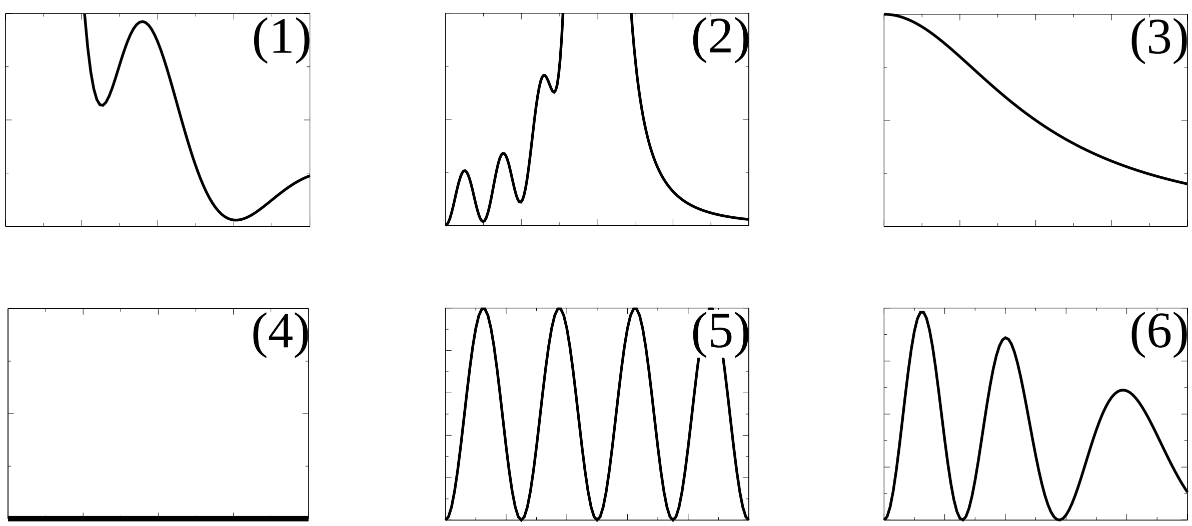
Two point sources at $x = 0$ and $y = \pm 2$ meters, emit radiation with $\lambda = 1$ meter. The two sources are out of phase by π, so the amplitude $A(x, y, t)$ of the waves at a point (x, y) a distance r_1 and r_2 from the two sources is $A_0(\cos(kr_1 - \omega t)/r_1 - \cos(kr_2 - \omega t)/r_2)$. (This question should look familiar from the last quiz in section.) From the figures (1-6) below, pick the one corresponding to the average intensity along the four dashed lines shown. The lower left corner in each plot is (0,0): five points each.)

(A) along $y = 0$, from $x = 0$ to $x = 4m$, ____________

(B) along $x = 0$, from $y = 0$ to $y = 4m$, ____________

(C) along $y = -2m$, from $x = 0$ to $x = 4m$, ____________

(D) at a distant screen $x = 100m$, from $y = 0$ to $y = 100m$. ____________



M2. (30 pts) Tensor Manipulations.

Let $\mathbf{a}$ and $\mathbf{b}$ be vector fields. For each of the formulas (A), (B), and (C), determine which *two* tensor formulas (1-6) are equal to it. (5 points each.)

(A) $\nabla(\mathbf{a} \cdot \mathbf{b})$ ______ ______

(B) $\nabla \cdot (\mathbf{a} \times \mathbf{b})$ ______ ______

(C) $\nabla \times (\mathbf{a} \times \mathbf{b})$ ______ ______

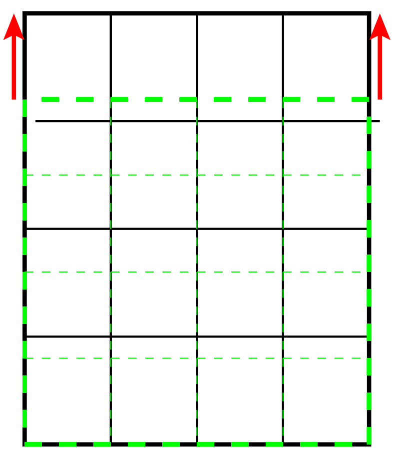
(1) $\epsilon_{ijk}\partial_i(a_j b_k)$
(2) $b_j\partial_i a_j + a_j\partial_i b_j$
(3) $\epsilon_{ijk}\epsilon_{k\ell m}\partial_j(a_\ell b_m)$
(4) $\partial_i(a_j b_j)$
(5) $b_i\epsilon_{ijk}\partial_j a_k - a_i\epsilon_{ijk}\partial_j b_k$
(6) $\partial_j(a_i b_j) - \partial_j(a_j b_i)$

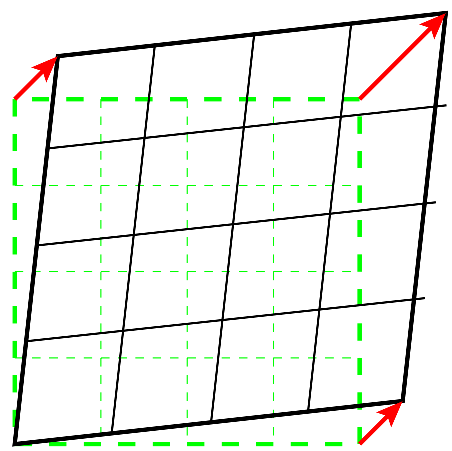
Check that you've put two choices for each of three formulas.

Short Answer: Show Your Work

S1. (50 pts) Elastic Properties of Square Crystals.

In this problem, we consider materials in two dimensions (x,y): you may ignore z or assume everything is uniform and undisplaced in the z direction. Let $\varepsilon^V = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_0 \end{pmatrix}$ and $\varepsilon^D = \begin{pmatrix} \varepsilon_0/2 & \varepsilon_0/2 \\ \varepsilon_0/2 & \varepsilon_0/2 \end{pmatrix}$ represent the elastic strains of two blocks of material, one strained vertically and the other strained along a diagonal, as shown in the figure below.





(A) If two points are separated by $\mathbf{d}^V = (0, a)$ in the undeformed state, what distance apart will they be in the deformed state with strain ε^V? (Hint: remember we derived ε so that two nearby points separated by $\mathbf{d}$ in the undeformed material had squared distance $d_i^2 + 2\varepsilon_{ij} d_i d_j$ in the deformed material.) Two points separated by $\mathbf{d}^D = (a, a)$ under ε^D? Do they have the same magnitude of elastic strain $\Delta L/L$ (with L vertical or diagonal distance, respectively), or not?

Distance for d^V under ε^V = ____________________

Distance for d^D under ε^D = ____________________

Same magnitude of strain? **T** **F**

(B) What would the elastic stress σ_{ij}^V be, corresponding to the vertical strain $\varepsilon^V = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon_0 \end{pmatrix}$, assuming an isotropic medium with elastic constants λ and μ? What would σ_{ij}^D be for the diagonal strain $\varepsilon^D = \begin{pmatrix} \varepsilon_0/2 & \varepsilon_0/2 \\ \varepsilon_0/2 & \varepsilon_0/2 \end{pmatrix}$?

$$\sigma_{ij}^V = \begin{pmatrix} \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \end{pmatrix} \qquad \sigma_{ij}^D = \begin{pmatrix} \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \\ \rule{2cm}{0.4pt} & \rule{2cm}{0.4pt} \end{pmatrix}$$

In lecture, we've focused on the elastic properties of isotropic materials, like glass and rubber, which act the same whichever direction they are strained. If our material is a crystal, with rows and columns of atoms along the $\hat{x}$ and $\hat{y}$ axes (like the grid points on the figure), it's clear that a vertical strain is different physically from a diagonal strain. (The diagonal deformation, for example, changes the bond angles, while the vertical one does not.)

Cubic crystals (or, in two dimensions, crystals with square symmetry) have three independent elastic constants: $c_{xxxx} = c_{yyyy} = C11$, $c_{xxyy} = c_{yyxx} = C12$ and $c_{xyxy} = c_{yxyx} = c_{xyyx} = c_{yxxy} = C44$, and the other tensor elements are zero. The constants $C11$, $C12$, and $C44$ are the names the engineers use for these three cubic elastic constants, which correspond to the two isotropic elastic constants λ and μ. (Capital C, big numbers, and only two indices distinguish them from the physics names for the elastic constants.)

(C) What is the elastic strain σ_{ij}^{CV} and σ_{ij}^{CD} for a square material with anisotropic elastic constants $C11$, $C12$ and $C44$, stretched vertically and diagonally as in part (B)?

$$\sigma_{ij}^{CV} = \begin{pmatrix} \rule{4cm}{0.4pt} & \rule{4cm}{0.4pt} \\ \rule{4cm}{0.4pt} & \rule{4cm}{0.4pt} \end{pmatrix} \quad \sigma_{ij}^{CD} = \begin{pmatrix} \rule{4cm}{0.4pt} & \rule{4cm}{0.4pt} \\ \rule{4cm}{0.4pt} & \rule{4cm}{0.4pt} \end{pmatrix}$$

Aluminum, a cubic crystal, happens to be nearly isotropic.

(D) Derive formulas for aluminum's three cubic elastic constants in terms of the two isotropic elastic constants λ and μ, using the isotropic formula $c_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$. You may wish to use it to check that your answers to parts (B) and (C) agree for aluminum.

$C11 =$____________

$C12 =$____________

$C44 =$____________

Physics 218—Exam II (November 9) Fall 2003

Formula Sheet

James P. Sethna

Sound Waves in Three Dimensions.

$\rho\partial^2\mathbf{u}/\partial t^2 = -\nabla p$, $p = -B\nabla\cdot\mathbf{u}$, $\partial^2 p/\partial t^2 = c^2\nabla^2 p$ with $c = \sqrt{B/\rho}$, $\partial^2\mathbf{u}/\partial t^2 = c^2\nabla^2\mathbf{u}$.
Spherical waves: $p(\mathbf{r},t) = f(|\mathbf{r}| - ct)/|\mathbf{r}|$.
Snell's law: $n_1\sin\theta_1 = n_2\sin\theta_2$, where the index of refraction $n = \sqrt{\epsilon\mu}$ is c/v.
Phase shift π on reflection where the index of refraction increases (*e.g.*, light off glass).
Intensity along the direction of propagation $I = p\partial\mathbf{u}/\partial t$,
energy density $E = (\rho/2)(\partial u/\partial t)^2 + p^2/(2B)$.
If $p(t) = \sum_n \tilde{p}_n \exp(i\omega_n t)$ and $\rho(t) = \sum_m \tilde{\rho}_m \exp(i\omega_m t)$, then the total power is the sum of the power in each frequency channel: $\sum_n (-i\omega/2)\tilde{p}_n\tilde{\rho}_n^*$.

Interference and Diffraction.

Double Slit. Phase difference $\phi = 2\pi d\sin(\theta)/\lambda = kd\sin(\theta)$.
Intensity $I_{av} = 4I_0\cos^2(\phi/2) = 4I_0\cos^2(kd\sin\theta/2)$ (I_0 single slit intensity).
Constructive for $d\sin\theta = 0, \pm\lambda, \pm 2\lambda, \ldots$, destructive for $d\sin\theta = \pm\lambda/2, \pm 3\lambda/2, \ldots$.
Multiple slits. $I_{av} = I_0\sin^2(N\phi/2)/\sin^2(\phi/2)$; principle maxima at $\phi = 0, 2\pi, 4\pi$, destructive at $\phi = 2m\pi/N$ with m any integer *except* $0, \pm N, \pm 2N, \ldots$.
Diffraction. If the slit opening is $f(x)$, $I_{av} \propto |\tilde{f}(k\sin\theta)|^2$.
The Fourier transform of a shifted function $f(x-\Delta)$ is $\exp(-i\Delta k)\tilde{f}(k)$.
Single wide slit. $I_{av} = I_{center}\sin^2\alpha/\alpha^2$ with $\alpha = ak\sin(\theta)/2$.

Tensor Notation.

Einstein convention: $a_{ijk\ell}b_{imno} = \sum_{i=1}^3 a_{ijk\ell}b_{imno}$.
Dot product $\mathbf{a}\cdot\mathbf{b} = a_i b_i$, matrix applied to vector $(M\mathbf{x})_i = M_{ij}x_j$, matrix multiplication $(MN)_{ij} = M_{ik}N_{kj}$, trace $Tr(M) = M_{ii}$.
Laplacian $\nabla^2 f = \partial_i\partial_i f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f$, divergence $\nabla\cdot\mathbf{v} = \partial_i v_i$.
Identity tensor δ_{ij}, equals one if $i = j$, zero otherwise.
Totally antisymmetric tensor $\epsilon_{ijk} : \epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kij}$. $\epsilon_{123} = 1 = \epsilon_{231} = \epsilon_{312}$, $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$, zero if any index repeats.
$(\mathbf{a}\times\mathbf{b})_i = \epsilon_{ijk}a_j b_k$, $(\nabla\times\mathbf{v})_i = \epsilon_{ijk}\partial_j v_k$, $\det M = (1/6)\epsilon_{ijk}\epsilon_{\ell mn}M_{i\ell}M_{jm}M_{kn}$.
$\delta_{ii} = 3$, $\epsilon_{ijk}\delta_{jk} = 0$, $\epsilon_{ijk}\epsilon_{ijk} = 6$, $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$, $\epsilon_{ijm}\epsilon_{k\ell m} = \delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}$.

Elasticity Theory.

Stress tensor $\sigma_{ij}\hat{\mathbf{n}}_j$ = Force/Area across surface perpendicular to $\hat{\mathbf{n}}$. $\sigma_{ij} = \sigma_{ji}$ because torques on small volumes must vanish. Force on a small volume $F_i = \partial_j\sigma_{ij}$. For hydrostatic pressure P, $\sigma_{ij} = -P\delta_{ij}$.
Strain tensor $\varepsilon_{ij} = (1/2)\left(\partial_i u_j + \partial_j u_i + \partial_i u_k\partial_j u_k\right)$, where the last term (the geometric nonlinearity) is usually ignored. Two nearby points separated by $\mathbf{d}$ in the undeformed material will have squared distance $d_i^2 + 2\varepsilon_{ij}d_i d_j$ in the deformed material. $\varepsilon_{ij} = \varepsilon_{ji}$. The strain tensor for uniform stretching $-\Delta V/V = 3\Delta L/L$ would be $\varepsilon_{ij} = (\Delta L/L)\delta_{ij}$.

Tensor of elasticity $c_{ijk\ell}$ gives Hooke's law for anisotropic media, $\sigma_{ij} = c_{ijk\ell}\varepsilon_{k\ell}$. $c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$. There are 21 possible independent elastic constants.
The elastic energy density $E = (1/2)\sigma_{ij}\varepsilon_{ij} = (1/2)c_{ijk\ell}\varepsilon_{ij}\varepsilon_{k\ell}$.
Isotropic moduli. The bulk modulus K is the same as B for fluids: $P = -K(\Delta V/V)$. Under a shear by an angle θ, $E = (1/2)\mu\theta^2$.
Under unconstrained stretching, $F/A = Y\Delta L/L$, and $\Delta W/W = -\nu\Delta L/L$, where ν is Poisson's ratio. $K = 2\mu/3 + \lambda$, $\nu = \lambda/2(\mu+\lambda)$, and $Y = (2\mu^2 + 3\lambda\mu)/(\mu+\lambda)$.
Isotropic Tensors. $c_{ijk\ell} = \mu(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) + \lambda\delta_{ij}\delta_{k\ell}$. $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$. $E = \mu\varepsilon_{ij}\varepsilon_{ij} + (\lambda/2)(\varepsilon_{kk})^2$.
Wave equations. $\rho_0\partial^2 u_i/\partial t^2 = \partial_j\sigma_{ij} = (1/2)c_{ijk\ell}\partial_j(\partial_k u_\ell + \partial_\ell u_k)$. For isotropic media, $\rho_0\partial^2 u_i/\partial t^2 = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i$, or $\rho_0\partial^2\mathbf{u}/\partial t^2 = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$.
Decomposing $\mathbf{u} = \mathbf{u}_T + \mathbf{u}_L$ with $\nabla\cdot\mathbf{u}_T = 0$ and $\nabla\times\mathbf{u}_L = 0$, we have $\partial^2\mathbf{u}_T/\partial t^2 = c_T^2\nabla^2\mathbf{u}_T$ and $\partial^2\mathbf{u}_L/\partial t^2 = c_L^2\nabla^2\mathbf{u}_L$, with $c_T = \sqrt{\mu/\rho_0}$ and $c_L = \sqrt{(\lambda+2\mu)/\rho}$.

WKB.

If $c(x)$ varies slowly, the solution to a wave equation $\partial^2\eta/\partial t^2 = c(x)^2\partial^2\eta/\partial x^2$ is approximately of the WKB form $\eta(x,t) = A(x)\cos((S(x) - \omega t))$, where the phase $dS(x)/dx = \omega/c(x) = k(x)$ and the amplitude $A(x) = A(x_0)\sqrt{c(x)/c(x_0)}$.

Formulas from Prelim I.

Trigonometry $f = \omega/2\pi$, and $k = 2\pi/\lambda$. $\exp(iz) = \cos(z) + i\sin(z)$, $\cos(z) = (\exp(iz) + \exp(-iz))/2$, and $\sin(z) = (\exp(iz) - \exp(-iz))/(2i)$.
Wave Equation Solutions. The wave equation

$$\partial^2\eta/\partial t^2 = c^2\partial^2\eta/\partial x^2$$

has a traveling wave solution $\eta(x,t) = f(x \pm ct)$, a standing-wave solution $\eta(x,t) = A\sin(kx)\sin(\omega t)$, and (as a special case) a traveling sine wave $\eta(x,t) = A\exp(i(kx - \omega t))$, where $\omega/k = c$.
Fourier Transform of a Gaussian. If $f(x) = (1/\sqrt{2\pi}\sigma)\exp\left(-x^2/2\sigma^2\right)$, $\tilde{f}(k) = \exp\left(-\sigma^2k^2/2\right)$.

Physics 218—Final Exam (December 16) **Fall 2003**
James P. Sethna

NAME:______________________________

S1 (40 pts) __________________

S2 (60 pts) __________________

S3 (40 pts) __________________

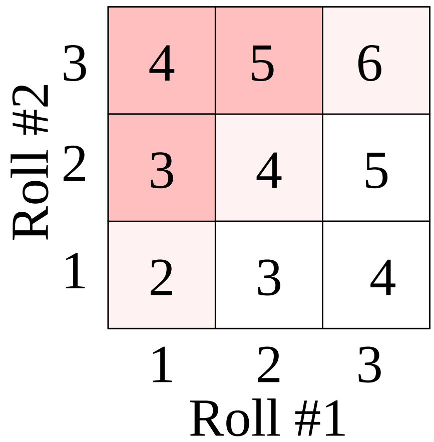
S4 (30 pts) __________________

S5 (30 pts) __________________

TOTAL (200 pts) __________________

Short Answer: Show Your Work

S1. (40 pts) Quantum Dice. (Thanks to Sarah Buchan.)



Rolling two dice. In *Bosons*, one accepts only the rolls in the shaded squares, with equal probability 1/6. In *Fermions*, one accepts only the rolls in the darkly shaded squares (not including the diagonal), with probability 1/3.

You are given several unusual 'three-sided' dice which, when rolled, show either one, two, or three spots. There are three games played with these dice, *Distinguishable*, *Bosons* and *Fermions*. In each turn in these games, the player rolls one of the dice at a time, starting over if required by the rules, until a legal combination occurs. In *Distinguishable*, all rolls are legal. In *Bosons*, a roll is legal only if the new number is larger or equal to the preceding number. In *Fermions*, a roll is legal only if the new number is strictly larger than the preceding number.

(A) Presume the dice are fair: each of the three numbers of dots shows up 1/3 of the time. For a legal turn rolling a die twice in *Bosons*, what is the probability $\rho(4)$ of rolling a 4? Similarly, among the legal Fermion turns rolling two dice, what is the probability $\rho(4)$?

Two rolls, three-sided Boson $\rho(4) =$ ____________________

Two rolls, three-sided Fermion $\rho(4) =$ ____________________

Comment: Our dice rules are the same ones that govern the quantum statistics of identical particles.

(B) For a legal turn rolling three 'three-sided' dice in *Fermions*, what is the probability $\rho(6)$ of rolling a 6?

Three rolls, three-sided Fermion $\rho(6) =$ ____________________

Hint: there's a Fermi exclusion principle: when playing Fermions, no two dice can have the same number of dots showing.
Comment: Electrons are fermions; no two electrons can be in exactly the same state.

When rolling two dice in *Bosons*, there are six different legal rolls (11), (12), (13), ..., (33): half of them are doubles (both numbers equal), when for plain old *Distinguishable* rolls only one third would be doubles*: the probability of getting doubles is enhanced by 1.5 times higher in two-roll *Bosons*. When rolling three dice in *Bosons*, there are ten different legal rolls (111), (112), (113), ..., (333). When rolling M dice each with N sides in *Bosons*, one can show that there are $\binom{N+M-1}{M} = \frac{(N+M-1)!}{M!(N-1)!}$ legal rolls.

(C) What is the enhancement of probability for 3 rolls of a three-sided dice of getting triples in *Bosons* over that in *Distinguishable*? What is the enhancement of probability for M rolls generating an M-tuple (all rolls having the same number of dots showing)?

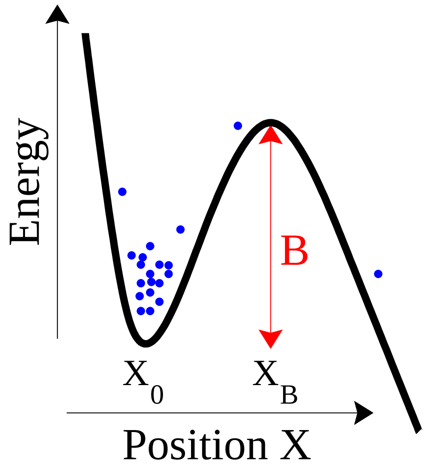
Enhancement for 3 rolls= ______________________

Enhancement for M rolls= ______________________

Comment: The states of the dice tend to clump together in *Bosons*. Examples of real bosons clumping into one state include superfluidity and lasers.

* For *Fermions*, of course, there are no doubles.

S2. (60 pts) Barrier Crossing.



Barrier crossing potential, and a schematic of how many atoms are at each position. (Actually, of course, the atoms are scattered at different X, not at different "heights" in energy.)

In this problem, we will derive the rate at which systems cross energy barriers, $\Gamma = \Gamma_0 \exp(-E/k_B T)$. We derived this Arrhenius formula in class, but did not carefully treat the prefactor Γ_0.

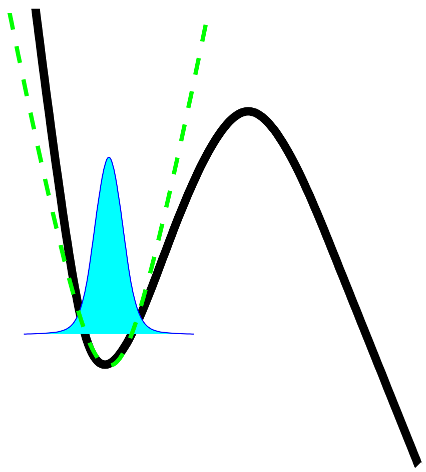
Consider a system described by a coordinate X, with an energy $U(X)$ with a minimum at X_0 with energy zero and an energy barrier at X_B with energy $U(X_B) = B$.* Let the temperature of the system be much smaller than B/k_B. To do our calculation, we will make some approximations. (1) We assume that the atoms escaping across the barrier to the right do not scatter back into the well. (2) We assume that the atoms deep inside the well are in equilibrium. (3) We assume that the particles crossing to the right across the barrier are given by the equilibrium distribution inside the well.

(A) Let the probability that a particle has position X be $\rho(X)$. What is the ratio of probability densities $\rho(X_B)/\rho(X_0)$ if the particles near the top of the barrier are assumed in equilibrium with those deep inside the well?

$\rho(X_B)/\rho(X_0) =$ ____________________

Related formulæ: Boltzmann distribution $\rho \propto \exp(-E/k_B T)$.

* This potential could describe a chemical reaction, with X being a reaction coordinate. It could describe the escape of gas from a moon of Jupiter, with X being the distance from the moon in Jupiter's direction.



Barrier crossing potential, and the approximate probability distribution for the atoms still trapped inside the well.

If the barrier height $B >> k_B T$, then most of the particles in the well stay near the bottom of the well. Often, the potential near the bottom is accurately described by a quadratic approximation $U(X) \approx (1/2)M\omega^2(X - X_0)^2$, where M is the mass of our system and ω is the frequency of small oscillations in the well.

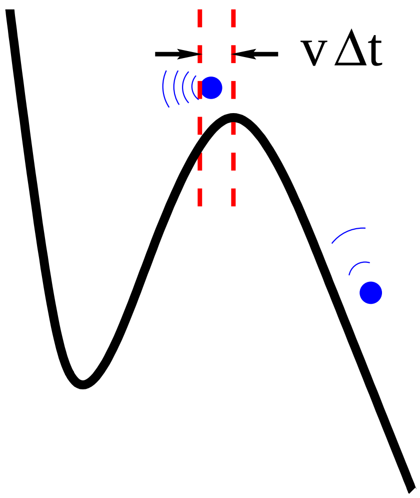
(B) In this approximation, what is the probability density $\rho(X)$ near the bottom of the well? What is $\rho(X_0)$, the probability density of having the system at the bottom of the well?

$\rho(X) =$________________________

$\rho(X_0) =$________________________

Related formulæ: Gaussian probability distribution $(1/\sqrt{2\pi\sigma^2}\exp(-x^2/2\sigma^2)$.

Hint: Make sure you keep track of the 2πs.



Barrier crossing potential, and a sketch of the range of positions that atoms moving to the right with velocity v will cross the barrier top in time Δt.

Knowing the answers from (A) and (B), we know the probability density $\rho(X_B)$ at the top of the barrier. We need to also know the probability that particles near the top of the barrier have velocity V, because the faster-moving parts of the distribution of velocities contribute more to the flux of probability over the barrier (see figure above). As we discussed in class, because the total energy is the sum of the kinetic and potential energy, the total Boltzmann probability factors: in equilibrium the particles will always have a velocity probability distribution $\rho(V) = 1/\sqrt{2\pi k_B T/M}\exp(-(1/2)MV^2/k_BT)$.

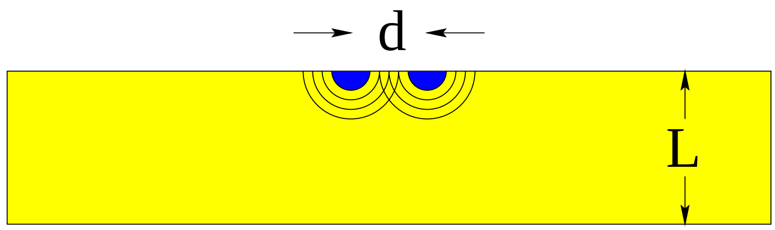
(C) First give a formula (for ease of grading) for the decay rate Γ (the probability per unit time that our system crosses the barrier towards the right), for an unknown probability density $\rho(X_B)\rho(V)$ as an integral over the velocity V. Then, using your formulas from parts (A) and (B), give our estimate of the decay rate for our system.

$\Gamma =$ ________________________ (integral form for unknown $\rho(X_B)$ and $\rho(V)$)

$\Gamma =$ ________________________ (using your answer from earlier parts).

Related formulæ: $\int_0^\infty x\exp(-x^2/2\sigma^2)\,dx = \sigma^2$.

S3. (40 pts) Interference of Elastic Waves.



An isotropic elastic solid of density ρ is cut into a slab of width L in the z direction. Two sources of vibration at frequency ω are attached to the top surface at $z = 0$, separated by a distance d in the y direction. The elastic moduli of the material are λ and μ. The two sources are hemispheres embedded into the material, which pulse outward with radius $r = r_0 + A\sin(\omega t)$, emitting compressive spherical waves into the medium.

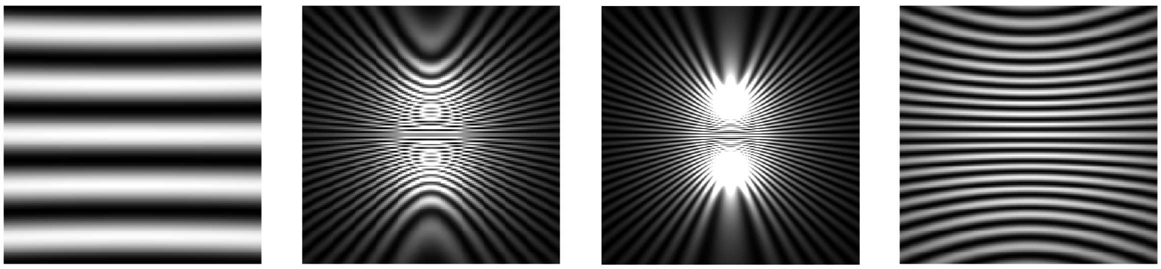
(A) If only one source is turned on, what wavevector k will the resulting spherical waves have?

$k =$ ______________________

Related formulæ: $c_T = \sqrt{\mu/\rho}$ for waves with zero divergence, $c_L = \sqrt{(\lambda + 2\mu)/\rho}$ for waves with zero curl.

The average sound intensity is measured at the bottom surface of the figure in part (A), as a function of x (horizontal in the figures below) and y (vertical) each ranging from (-4m, 4m). (Ignore reflections from the surfaces.) The sources are at $y = \pm 1$ m, and the wave vector of the sound is $k = 50$. The experiment is repeated four times, with different slab thicknesses L. The grayscales in the different figures are not the same, but the black and white regions do represent local constructive and destructive interference.

(B) Which figure corresponds to which distance?



(1) (2) (3) (4)

$L = 0.2$ is figure ____________

$L = 1$ is figure ____________

$L = 5$ is figure ____________

$L = 25$ is figure ____________

Related formulæ: Spherical wave amplitude $A = Re[\exp\left(i(kr - \omega t)\right)/r$; two sources $d \sin\theta = n\lambda$ for constructive interference.

S4. (30 pts) Diffusion Tensor.

Consider diffusion in an anisotropic material like graphite. Atoms stuck between layers will diffuse relatively easily in the plane parallel to the layers, but there will be very little diffusion perpendicular to the layers.

In general, the diffusion tensor D_{ij} relates the (vector) current J_i to the (vector) gradient of the density $\partial_j \rho$:

$$J_i = -D_{ij} \partial_j \rho \tag{S4.1}$$

The diffusion tensor is symmetric, $D_{ij} = D_{ji}$.

(A) Given the definition (S4.1) above, derive the anisotropic diffusion equation of motion for $\partial\rho/\partial t$.

$\partial\rho/\partial t =$ ______________________________

Related formula: Conservation of probability: $\partial\rho/\partial t = -\nabla \cdot J$.

A piece of graphite has layers parallel to the xy plane. An atom starts at the origin at $t = 0$ and undergoes a random walk through the graphite. At equally spaced times $t = n\tau$ the atom takes a step from (x, y, z) to one of the four positions $(x \pm a, y, z)$ or $(x, y \pm a, z)$, each with probability 1/4. In addition, at times $t = 200m\tau$ the atom takes a step to $(x, y, z \pm c)$, each with probability 1/2.

(B) After a time $t = T$, what will the root-mean-square distance moved be by the atom along x, along y, and along z?

$\sqrt{\langle x^2 \rangle} =$ ______________________________

$\sqrt{\langle y^2 \rangle} =$ ______________________________

$\sqrt{\langle z^2 \rangle} =$ ______________________________

Related one-dimensional formulæ: If $s_N = \sum_{i=1}^{N} x_i$ and $\langle x_i x_j \rangle = \sigma^2 \delta_{ij}$, then $\langle s_N^2 \rangle = \langle s_{N-1}^2 \rangle + \sigma^2 = N\sigma^2$.

The diffusion tensor for graphite is diagonal: only the three entries D_{11}, D_{22}, and D_{33} are non-zero. One can show that the root-mean-squared distance diffused along the three axes is given by the corresponding diffusion constant, so for example $\langle y^2 \rangle = 2D_{22}t$.

(C) Fill in the values for D_{ij} for the continuum limit of the random walk described in part (B).

$$D_{ij} = \begin{pmatrix} \rule{3cm}{0.4pt} & 0 & 0 \\ 0 & \rule{3cm}{0.4pt} & 0 \\ 0 & 0 & \rule{3cm}{0.4pt} \end{pmatrix}$$

Related one-dimensional formulæ: $\sqrt{\langle x^2 \rangle} = \sqrt{2Dt}$; $G(x,t) = 1/\sqrt{4\pi Dt}\, \exp(-x^2/4Dt)$

S5. (30 pts) Information entropy.

Entropy is a measure of your ignorance about a system: it is a measure of the lack of information. It has important implications in communication technologies: messages passed across the Ethernet communicate information, reducing the information entropy for the receiver. Shannon worked out the use of entropy ideas in communications, focusing on problems where different messages have different probabilities. We'll focus on the simpler problem where all N messages are equally likely. Shannon defines the *information entropy* of an unread message as being $\log_2 N = k_S \log N$, where $k_S = 1/(\log_e 2)$ is analogous to Boltzmann's constant, and changes from log-base-e to log-base-2 (more convenient for computers, which think in base two.)

Your grandparent has sent you an e-mail message. From the header of the message, you know it contains 1000 characters. You know each character is made of 8 bits, which allows $2^8 = 256$ different letters or symbols per character. Assuming all possible messages from your grandparent are equally likely (a typical message would then look like G*(me'!8V[beep]...), how many different messages N could there be? This (unrealistic) assumption gives an upper bound for the information entropy S_{max}.

(A) What S_{max} for the unread message?

$N =$ ____________________

$S_{max} =$ ____________________

Your grandparent writes rather dull messages: they all fall into the same pattern. They have a total of 16 equally likely messages. * After you read the message, you forget the details of the wording anyhow, and only remember these key points of information.

(B) What is the actual information entropy change $\Delta S_{Shannon}$ you undergo when reading the message? If your grandparent writes one message per month, what is the minimum number of 8-bit characters per year that it would take to send your grandparent's messages? (You may lump multiple messages into a single character.)

$\Delta S_{Shannon} =$ ______________________

Characters per year = ______________________

Hints: $\Delta S_{shannon}$ is the change in entropy from before you read the message to after you read which of 16 messages it was. The length of 1000 is *not* important for this part.
Remark: This is an extreme form of data compression, like that used in gif images, zip files (Windows) and gz files (Unix). We are asking for the number of characters per year for an optimally compressed signal.

* Each message mentions whether they won their bridge hand last week (a fifty-fifty chance), mentions that they wish you would write more often (every time), and speculates who will win the women's college basketball tournament in their region (picking at random one of the eight teams in the league.