FOURIER SERIES

Superposition Gone Berserk

Math Truth: "Any" periodic function $y(x)$ with period L can be written as a sum of sinusoidal functions with the same period.

Sine and Cosine Series:

$$y(x) = \sum_{m=0}^{\infty} a_m \cos(k_m x) + b_m \sin(k_m x)$$

$\cos(k_m x) = \cos(k_m(x+L))$ if $k_m L = 2\pi m \Rightarrow \boxed{k_m = \frac{2\pi m}{L}}$

(Not the same as $\overset{y(0)=}{y(L)} = 0 \Rightarrow a_m = 0,\ k_m = \frac{\pi m}{L}$)

If we write $\cos(k_m x) = \frac{e^{ik_m x} + e^{-ik_m x}}{2}$, $\sin(k_m x) = \frac{e^{ik_m x} - e^{-ik_m x}}{2i}$
$= -\frac{i}{2} e^{ik_m x} + \frac{i}{2} e^{-ik_m x}$

Complex Fourier Series:

$$y(x) = \sum_{m=-\infty}^{\infty} \tilde{y}_m e^{ik_m x}$$

$$\tilde{y}_m = \begin{cases} a_m/2 - i\, b_m/2 & m > 0 \\ a_0 & m = 0 \\ a_m/2 + i\, b_m/2 & m < 0 \end{cases}$$

Complex Conjugates.
b_0 = undefined, $\sin(0x) \equiv 0$

The complex Fourier series of a real function $y(x)$ satisfies $\tilde{y}_{-m} = \tilde{y}_m^*$

Why Bother with Fourier Series?

(1) Humans Perceive Frequencies

Fourier series in <u>time</u>:

Pressure $P(t) = \sum \tilde{p}(\omega) e^{i\omega t}$

High Pitch $\Leftrightarrow$ Big ω $\qquad f = \omega/2\pi \sim 20\text{–}20{,}000\ Hz$

Low Pitch $\Leftrightarrow$ Small ω

Light $\vec{E}(t) = \sum \tilde{\vec{E}}(\omega) e^{i\omega t}$ $\qquad$ (Hz = Hertz = cycles/sec, ω in radians/sec)

Blue = High Frequency $\qquad f \sim 5\times10^{16}$ Hz

Red = Low Frequency

(2) Sinusoids solve Equations

Wave Equation, $y(x,0) = \sin(k_m x)$, $\frac{\partial y}{\partial t} = 0$ at $t=0$

$\rightarrow$ Standing Wave $\quad y(x,t) = \sin(k_m x)\cos(\omega_m t)$

$$\omega_m = c k_m = \frac{2\pi m}{L} c$$

so, by superposition, we can solve for <u>any</u> initial shape

$$y(x,0) = \sum b_m \sin(k_m x)\ , \quad \frac{\partial y}{\partial t}(x, t=0) = 0$$

$$\Rightarrow y(x,t) = \sum b_m \sin(k_m x)\cos\omega_m t$$

(Not so necessary for wave equation, which has special traveling wave solutions that are much easier: Elmore & Heald use Fourier methods, we don't. Will be crucial in the diffusion equation later.)

(3) Derivatives are Easy

$$\frac{d}{dx} e^{ikx} = ik e^{ikx} \quad \text{(Derivative} \Rightarrow \text{multiply by } ik\text{)}$$

$$\frac{d}{dx} y(x) = \sum_m \tilde{y}_m \frac{d}{dx}\left(e^{ik_m x}\right) = \sum_m \underbrace{\left(ik_m \tilde{y}_m\right)}_{m^{th} \text{ Fourier coefficient of } dy/dx} e^{ik_m x}$$

Example:

$$y(x,t) = e^{ikx} e^{i\omega t}$$

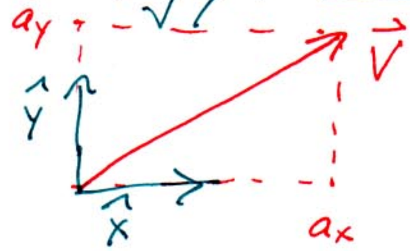
$$\frac{\partial^2 y}{\partial t^2} = (i\omega)^2 e^{ikx} e^{i\omega t} = c^2 \frac{\partial^2 y}{\partial x^2} = (ik)^2 c^2 e^{ikx} e^{i\omega t}$$

$$\Rightarrow \omega^2 = c^2 k^2$$

(Replaces $\frac{d \sin\theta}{d\theta} = \cos\theta$, $\frac{d\cos\theta}{d\theta} = -\sin\theta$
for cosine/sine transform)

How do you find $\tilde{y}_m$?

Analogy: <u>Vectors in 2D</u>



$$\vec{V} = a_x \hat{x} + a_y \hat{y}$$

$$a_x = \vec{V} \cdot \hat{x} \qquad a_y = \vec{V} \cdot \hat{y}$$

unit vectors $\hat{x}, \hat{y}$
orthogonal
"Basis vectors"

Self consistent?

$$a_x \overset{?}{=} \vec{V} \cdot \hat{x} = (a_x \hat{x} + a_y \hat{y}) \cdot \hat{x}$$

$$= a_x \underbrace{\hat{x} \cdot \hat{x}}_{\substack{\text{Normalized} \\ =1}} + a_y \underbrace{\hat{y} \cdot \hat{x}}_{\substack{\text{Orthogonal} \\ =0}} \overset{\checkmark}{=} a_x$$

(1) <u>Any</u> vector $\vec{V}$ can be written as sum over basis vectors

(2) Coefficients given by dot products <u>if</u> orthonormal basis.

FUNCTION SPACE

Space of functions with period L

Basis functions $e^{ik_m x}$ $m = \ldots -2, -1, 0, 1, \ldots$

Infinite dimensional space
(Loosely, one "dimension" for each point $x \in (0, L]$)

Dot Product in Function Space

$$f \cdot g = \frac{1}{L}\int_0^L f(x)\, g^*(x)\, dx \quad \Big\} \text{Like } V \cdot W = V_x W_x + V_y W_y + V_z W_z$$

Because our functions are complex, need complex conjugate of second function, so $f \cdot f > 0$

so $f \cdot f = \frac{1}{L}\int_0^L f(x) f^*(x)\, dx = \frac{1}{L}\int_0^L |f(x)|^2 dx > 0$

Is our basis normalized? Is $y(x) = e^{ik_n x}$ dot with itself equal to one?

$$\frac{1}{L}\int_0^L \underbrace{e^{ik_n x}\, e^{-ik_n x}}_{=1}\, dx = \frac{1}{L} \cdot x \Big|_0^L = 1 \checkmark$$

Is our basis orthogonal? Is $y(x) = e^{ik_n x}$ dot with $e^{ik_m x}$ equal to zero, if $m \neq n$?

$$\frac{1}{L}\int_0^L e^{ik_n x}\, e^{-ik_m x}\, dx = \frac{1}{L}\int_0^L e^{i(k_n - k_m)x}\, dx$$

$$= \frac{1}{L(ik_n - k_m)}\Big(\underbrace{e^{i\overbrace{(k_n - k_m)L}^{2\pi(n-m)}}}_{1} - 1\Big) = 0 \checkmark$$

Self Consistent?

$$\tilde{y}_\ell \overset{?}{=} \frac{1}{L}\int y(x)\, e^{-ik_\ell x}\, dx$$

$$= \frac{1}{L}\int \sum \tilde{y}_m e^{ik_m x} e^{-ik_\ell x}\, dx$$

$$= \sum_m \tilde{y}_m \underbrace{\frac{1}{L}\int e^{ik_m x} e^{-ik_\ell x}\, dx}_{\substack{\text{One if } m=\ell \\ \text{Zero otherwise} \\ \text{Orthonormal!}}} = \tilde{y}_\ell \checkmark$$