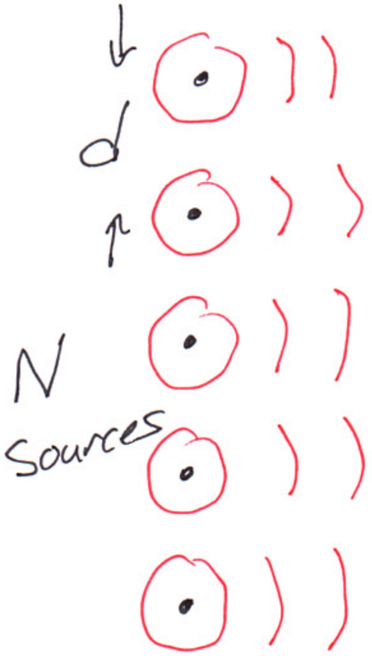
Multiple Slits / Gratings / Antenna Arrays / Bragg Scattering



Multiple sources of waves, spaced by d

Demo: Diffraction off a CD ROM

- 6000 lines / cm
- N = spot size / grid spacing
- Note zeroth order, first order, second-order peaks

Phase shift between waves $\phi = \frac{2\pi}{\lambda} d \sin\theta$

$$= k d \sin\theta$$

Amplitude $E(\theta) \propto \sum_{n=0}^{N-1} \cos(\omega t + n\phi)$

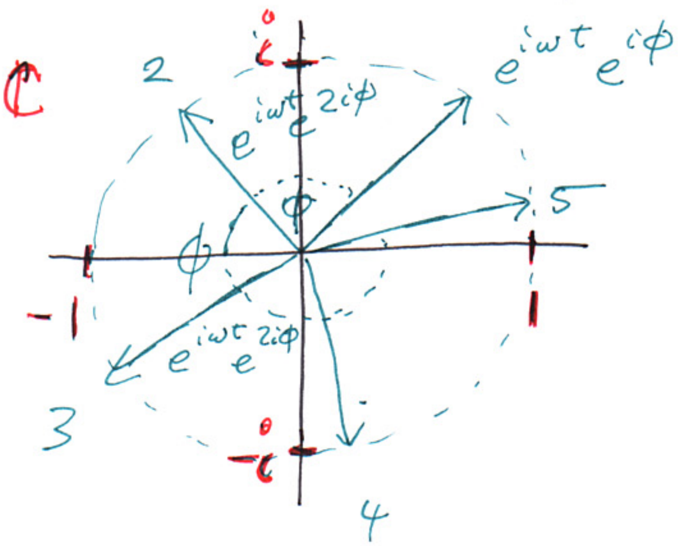
$$= \mathrm{Re}\left(\sum_{n=0}^{N-1} e^{i(\omega t + n\phi)}\right)$$

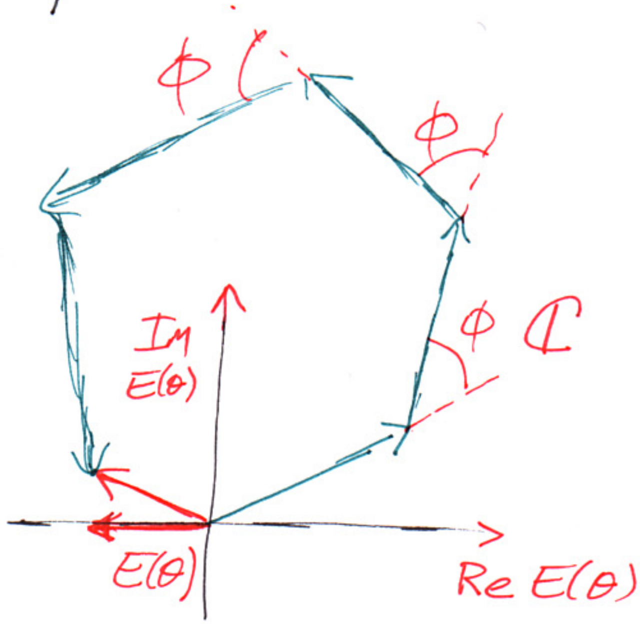
$$= \mathrm{Re}\left(e^{i\omega t} \sum_{n=1}^{N-1} \left(e^{i\phi}\right)^n\right)$$

$$(1 + x + x^2 + \dots + x^{N-1})(1-x) = 1 - x^N$$

$$= \mathrm{Re}\left(e^{i\omega t} \frac{1-e^{iN\phi}}{1-e^{i\phi}}\right)$$

$$E(\theta) = \mathrm{Re}\left(e^{i\omega t}\sum_{n=0}^{N-1}\left(e^{i\phi}\right)^{n}\right)$$





"Phasor" Diagram: Vector Sum of Complex Amplitudes

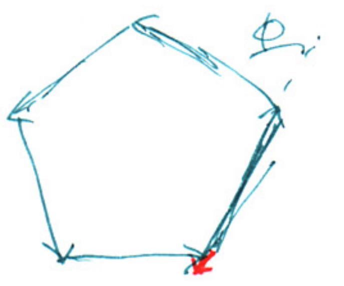
Geometry

$I(t) = \varepsilon_0 c E^2(t)$

Square of Length of Sum Vector ∝ Net Intensity

Main Peaks $\phi = 0, \pm 2\pi, \ldots$

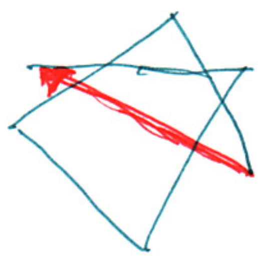
$I_{av} = N^2 I_{source}$



First Minimum

$N\phi = 2\pi$ = Total Angle Rotated

$\phi = 2\pi/N$



Second "Maximum" $\phi = \frac{3\pi}{N}$

Second Minimum  $\phi = \frac{4\pi}{N}$

Algebra

$$E(\theta) = \mathrm{Re}\left(e^{i\omega t} \sum_{n=0}^{N-1} (e^{i\phi})^n\right)$$

$$= \mathrm{Re}\left(e^{i\omega t} \frac{1-e^{iN\phi}}{1-e^{i\phi}}\right)$$

$$= \mathrm{Re}\left[e^{i\omega t}\left(\frac{+e^{iN\phi/2}}{e^{i\phi/2}}\right)\left(\frac{e^{iN\phi/2}-e^{-iN\phi/2}}{e^{i\phi/2}-e^{-i\phi/2}}\right)\right]$$

$$= \mathrm{Re}\left[e^{i(\omega t+(N-1)\phi/2)} \frac{\sin(N\phi/2)}{\sin(\phi/2)}\right]$$

$I(t) = \varepsilon_0 c E^2(t)$

$$I_{av} \propto \frac{1}{T}\int_0^T E^2(\theta)\, dt = \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)} \underbrace{\langle \cos^2(\omega t + (N-1)\phi/2)\rangle}_{1/2}$$

($= \varepsilon_0 c$)

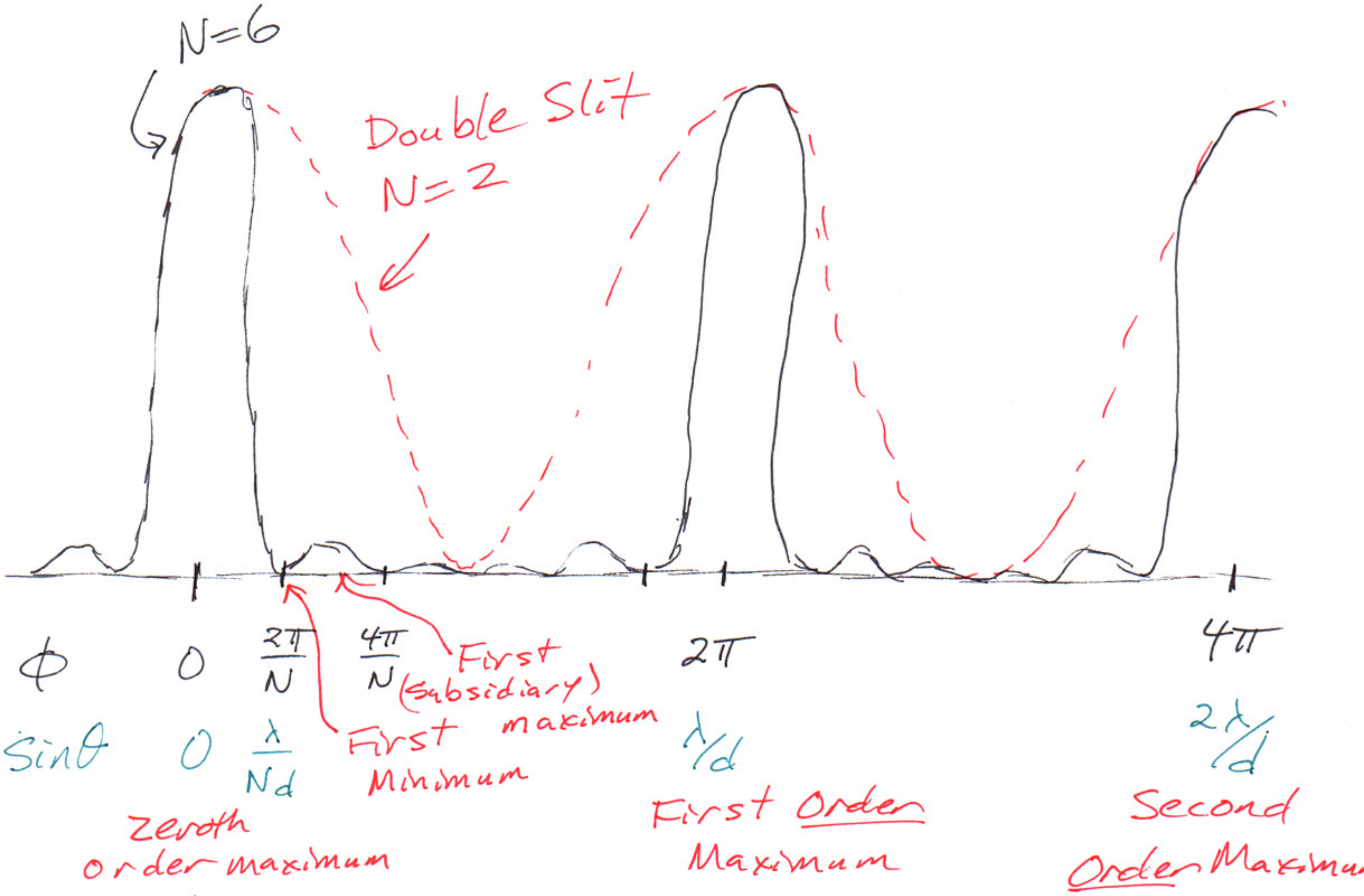
$$\frac{I_{av}(\theta)}{I_{av}(\text{one source})} = \frac{\sin^2(N\phi/2)}{\sin^2(\phi/2)}$$

$\theta = 0 \rightarrow \frac{0}{0}$, Taylor expand $\approx \left(\frac{N\phi/2}{\phi/2}\right)^2 = N^2$ ✓

$\phi = kd\sin\theta = \frac{2\pi}{N} \rightarrow \frac{0}{\sin^2(\pi/N)} = 0$ minimum ✓

$\phi = \frac{3\pi}{N} \rightarrow \ldots \quad \phi = 2\pi \rightarrow \frac{\sin^2\left(\frac{2\pi N}{2}\right)}{\sin^2(\pi)} = \frac{0}{0} \ldots N^2$ ✓

Periodic $\phi \rightarrow \phi + 2\pi$



Notice!

- Big $N \Rightarrow$ Sharper Peaks at $d \sin\theta = N\lambda$
- Use position of peak to measure wavelength (Diffraction grating spectrometer)
- Zeros at $\frac{2m\pi}{N}$ <u>except</u> $m = 0, \pm N, \ldots$