Tensors

Remember Maxwell's equations in matter:

$$\vec{\nabla}\cdot\vec{D} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J}$$
$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\,\partial\vec{B}/\partial t = 0$$
$$\vec{\nabla}\cdot\vec{B} = 0$$

(1) Maxwell originally wrote out each component of these equations [8 equations in all]. Vector notation helps a lot. Other equations involve <u>tensors</u>: where vectors have one index E_i, tensors can have two or more ε_{ij}, C_{ijkl}, ...

(2) We can write Maxwell's equations in a more modern "tensor" notation:

$$\partial_i D_i = 4\pi\rho$$

$\partial_i = \partial/\partial x_i$
Einstein convention $x\cdot y = x_i y_i = \sum_{i=1}^{3} x_i y_i$
drop summation sign

$$\varepsilon_{ijk}\nabla_j H_k - \frac{1}{c}\partial_0 D_i = \frac{4\pi}{c}J_i$$

$\partial_0 = \frac{\partial}{\partial t}$ (high energy)
~~Roman~~ index 1,2,3
Greek 0,1,2,3 (high energy)

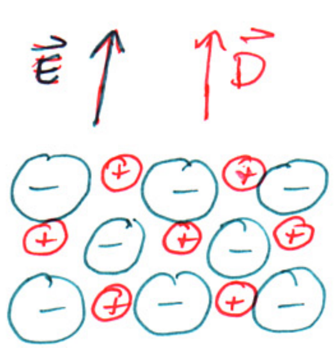
$$\varepsilon_{ijk} = \begin{cases} 0 & \text{any index repeats} \\ 1 & \varepsilon_{123}, \varepsilon_{231}, \varepsilon_{312} \\ -1 & \varepsilon_{132}, \varepsilon_{321}, \varepsilon_{213} \end{cases}$$

ε_{ijk} = Totally antisymmetric tensor in three indices
= Levi C_____ tensor

Exercise: Write the other two in modern notation

$$\varepsilon_{ijk}\partial_j E_k + \frac{1}{c}\partial_0 B_i = 0 \qquad \partial_i B_i = 0$$

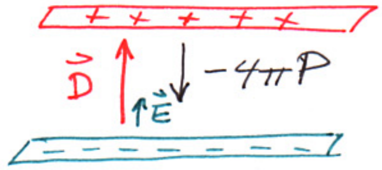
(3) What is D?



Vacuum: $\vec{D} = \vec{E}$

NaCl

Na shift up $\uparrow$ Cl shift down $\downarrow$



Atomic Polarization

Both cases, net polarization charge moves up

$P \uparrow$

Equivalent to Charge layer on top & bottom

In matter, $\vec{E}$ is the electric field averaged over the crystal unit cell. It includes a component $-4\pi \vec{P}$ due to the polarization of the crystal. ~~Both~~

$$\vec{D} = \vec{E} + 4\pi \vec{P} \qquad \text{MKS: } \vec{D} = \vec{E} - \frac{\vec{P}}{\varepsilon_0}$$

$$= \epsilon \vec{E}$$

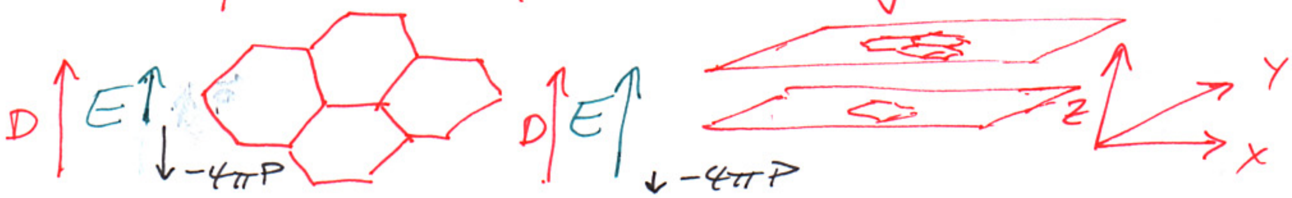
where ε is the dielectric constant. Also,

$$P = \alpha E$$

where α is the polarizability, $\varepsilon = 1 + 4\pi\alpha$

(4) Dielectric constant ε (and α) are <u>tensors</u>, for complex crystals.

Graphite: layers of hexagons



Polarizability (probably) much higher in the plane than perpendicular to planes

$$\alpha_{ij} = \begin{pmatrix} (\text{Big}) & 0 & 0 \\ 0 & (\text{Big}) & 0 \\ 0 & 0 & (\text{small}) \end{pmatrix}$$

$$P_i = \alpha_{ij} E_j = \sum_{j=1}^{3} \alpha_{ij} E_j = \underset{\approx}{\alpha} \cdot \vec{E}$$

Matrix application to vector

$$D_i = \varepsilon_{ij} E_j$$

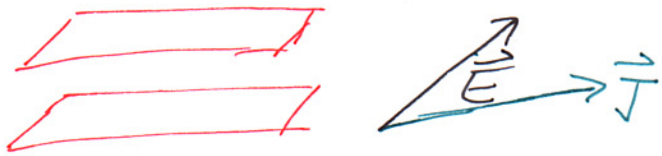
Other Tensors

Conductivity

$$j_m = \sigma_{mn} E_n$$

can be asymmetric in magnetic field: Hall effect

Graphite $\sigma = \begin{pmatrix} \text{Big} & 0 & 0 \\ 0 & \text{Big} & 0 \\ 0 & 0 & \text{small} \end{pmatrix}$



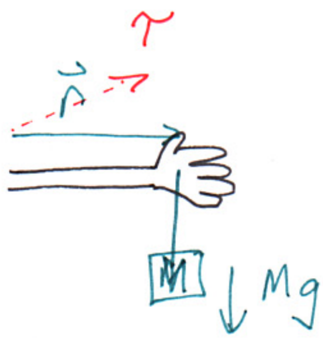
Notice: $\vec{J}, \vec{E}$ not parallel in general

Moment of Inertia

(Feynman ...)

¿Torque?

$$\vec{\tau} = \vec{r} \times \vec{F}$$



What is the deep inner meaning of $\vec{\tau}$ pointing into the board?

Right-hand rule convention?

Tensor Notation

$$\tau_i = \varepsilon_{ijk} r_j F_k = \frac{1}{2} \varepsilon_{ijk} (r_j F_k - r_k F_j)$$

$$= \frac{1}{2} \varepsilon_{ijk} \begin{pmatrix} 0 & \tau_z & -\tau_y \\ -\tau_z & 0 & \tau_x \\ \tau_y & -\tau_x & 0 \end{pmatrix}$$

Torque is "really" an antisymmetric 3×3 matrix! In three dimensions, we use ε_{ijk} to change it into a vector.

$\det M = \varepsilon_{ijk} \varepsilon_{lmn} M_{il} M_{jm} M_{kn}$ $\quad (M \cdot N)_{ik} = M_{ij} N_{jk}$