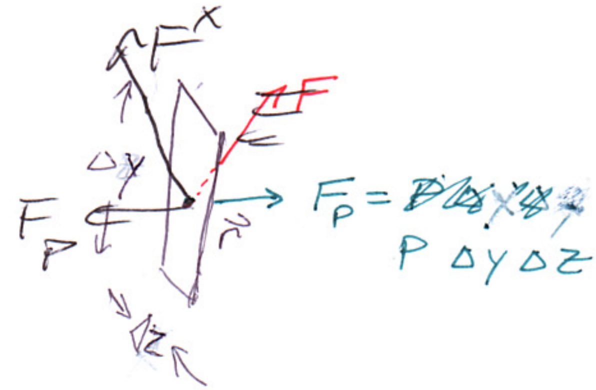
Add K, Y, σ, μ

Tensors of Elasticity

The Stress Tensor



Consider a small planar region $\Delta x \Delta z$ passing through a point $\vec{r}$. What is the force of the material to the left of $\vec{r}$ on the material to the right?

Proportional to $\Delta y \Delta z$: divide out

Define

$$\sigma_{i1} = F_i^x / \Delta y \Delta z$$

$$\sigma_{i2} = F_i^y / \Delta z \Delta x$$

$$\sigma_{i3} = F_i^z / \Delta x \Delta y$$

Exercise: If the material is a fluid, at pressure P, what is the force?

$$\vec{F} = P \Delta y \Delta z \, \hat{x}$$

$$\frac{F_1}{\Delta y \Delta z} = \sigma_{11} = P$$

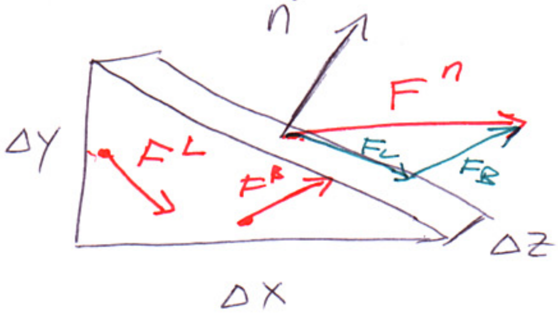
$$F_2 = F_3 = 0$$

σ_{ij} = Stress Tensor

$$\tfrac{1}{3}\sigma_{ii} = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 = \text{Pressure}$$

<u>Fact</u> $\sigma_{ij}\hat{n}_j = \left(\frac{\text{Force}}{\text{Area}}\right)$ across surface perpendicular to $\hat{n}$

Proof in Two Dimensions:



Assume tiny triangle of material in equilibrium: force

$$\vec{F}^L + \vec{F}^B = \vec{F}^n$$

(As size → 0, mass → 0, acceleration finite → net force = 0)

$$\hat{n} = \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}}\begin{pmatrix}\Delta y \\ \Delta x\end{pmatrix}$$

$$\vec{F}_i^L = \Delta y \Delta z \; \sigma_{i1}$$

$$F_i^B = \Delta x \Delta z \; \sigma_{i2}$$

$$\vec{F}_i^n = \sigma_{i1}\Delta y \Delta z + \sigma_{i2}\Delta x \Delta z$$

Area of Face $= \Delta z \sqrt{\Delta x^2 + \Delta y^2}$

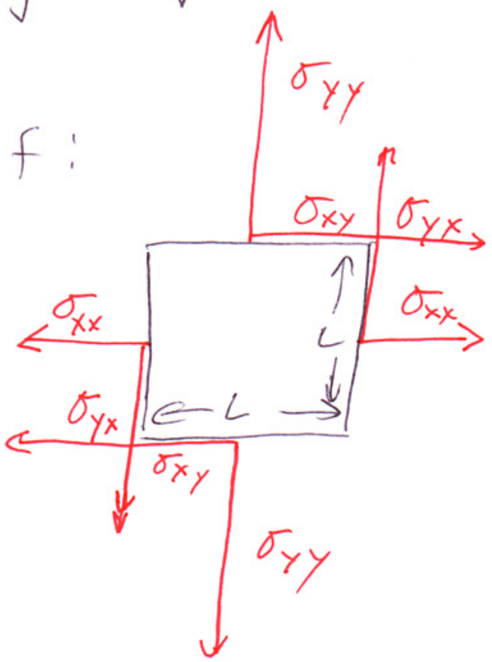
$$\frac{\vec{F}_i^n}{\text{Area}} = \sigma_{i1}\frac{\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} + \sigma_{i2}\frac{\Delta x}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$= \sigma_{i1}\hat{n}_1 + \sigma_{i2}\hat{n}_2 = \sigma_{ij}\hat{n}_j \checkmark$$

Fact

$\sigma_{ij} = \sigma_{ji}$ (Stress is a symmetric tensor)

Proof:



Assume cube is in equilibrium

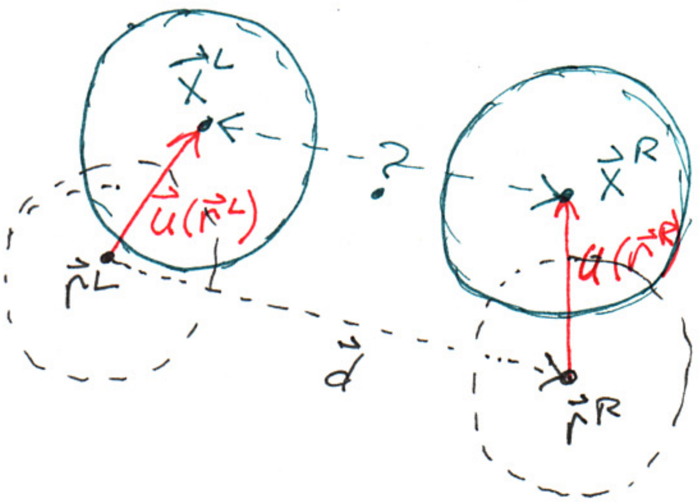
Total Force $= 0$ ✓

Total Torque
$= (\sigma_{yx} - \sigma_{xy}) \cdot L = 0$

$\Rightarrow \sigma_{xy} = \sigma_{yx}$

The Strain Tensor

[Standard derivation in texts; this fancy one from Landau & Lifshitz ~~Elasticity~~ *Theory of Elasticity* p. 1]



Two atoms, undistorted positions $\vec{r}^L$, $\vec{r}^R$, distorted positions $\vec{X}^L$, $\vec{X}^R$. Displacement field

$$\vec{u}(\vec{r}) = \vec{X}(\vec{r}) - \vec{r}.$$

Original distance $\vec{d}^2 = (\vec{r}^R - \vec{r}^L)^2$. Final distance?

Strain tensor tells how far the original distance vector $\vec{d}$ will stretch.

$$(\vec{X}^R - \vec{X}^L)^2 = [\vec{r}^R + \vec{u}(\vec{r}^R) - \vec{r}^L - \vec{u}(\vec{r}^L)]^2$$

$$= [d_i + u_i(\vec{r}^L + \vec{d}) - \vec{u}_i(\vec{r}^L)]^2$$

Summation convention $a_i^2 = a_i a_i = \sum_{i=1}^{3} a_i^2$

$$\cong [d_i + d_j \partial_j u_i]^2$$

$\vec{u}(\vec{x}+\vec{h}) - \vec{u}(\vec{x}) \approx (\vec{h}\cdot\vec{\nabla})\vec{u}$ true for vectors as well as functions

Old distance

$$= d_i^2 + 2 d_i d_j \partial_j u_i + d_j \partial_j u_i d_k \partial_k u_i$$

$$d_i d_j \partial_j u_i = d_i d_j \partial_i u_j = d_i d_j \left(\frac{\partial_j u_i + \partial_i u_j}{2}\right)$$

Interchange dummy indices i and k

$$(\text{New distance})^2 - (\text{Old distance})^2$$

$$= 2\, d_i d_j \left[\frac{\partial_i u_j + \partial_j u_i}{2} + \frac{\partial_i u_k \partial_j u_k}{2} \right]$$

$$= 2\, d_i d_j\, \varepsilon_{ij}$$

Strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\Big[\underbrace{\partial_i u_j + \partial_j u_i}_{\text{Symmetrized Gradient}} + \overbrace{\partial_i u_k \partial_j u_k}^{\text{Geometric Nonlinearity}} \Big]$$

The geometric nonlinearity is usually ignored. Strains for most materials (not rubber) above 1% lead to permanent bending (plastic flow).

$$\varepsilon_{ij} \sim 0.01 \Rightarrow \partial_i u_k \partial_j u_k \sim 10^{-4}, \text{ ignore.}$$

<u>Except</u>: Rubber

Large rotations (wires, thin sheets, grain boundaries)

The Tensor of Elasticity

Hooke's Law $F = KX$, $F/A = Y \frac{\Delta L}{L}$, ...

Force ⇔ Stress tensor σ_{ij}

Stretch ⇔ Strain tensor ε_{kl}

$$\sigma_{ij} = \underbrace{C_{ijkl}}_{\text{Elastic Tensor}} \varepsilon_{kl}$$

- How many elastic constants for an anisotropic material?

i, j, k, l 1,2,3 ⇒ 81 at most.

$\varepsilon_{kl} = \varepsilon_{lk} \to (C_{ijkl} - C_{ijlk})$ value irrelevant for σ

$\to$ set $C_{ijkl} = C_{ijlk}$

$C_{ijkl} = C_{jikl}$

Six choices of $\{ij\}$, six choices of $\{kl\}$

$\to$ 36 at most.

Use formula for the energy, ~~Problem 7.6.21~~

Problem 7.4.1 ⇒ 21 possible independent elastic constants.