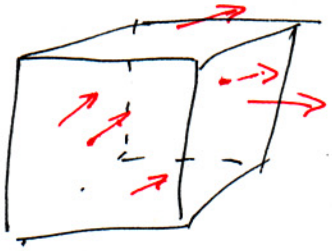
Forces

Net Force on a Small Cube:



$$F_i = \sum_{6\text{ faces}} \left(i^{\text{th}}\text{-component of force on face} \right)$$

$$= \left[\sigma_{i1}(x+\Delta x) - \sigma_{i1}(x)\right]\Delta y \Delta z$$

$$+ \left[\sigma_{i2}(y+\Delta y) - \sigma_{i2}(y)\right]\Delta x \Delta z$$

$$+ \left[\sigma_{i3}(z+\Delta z) - \sigma_{i3}(z)\right]\Delta x \Delta y$$

$$\boxed{F_i = \partial_j \sigma_{ij}\, dV}$$

Newton's Law:

$$F = ma = m\frac{\partial \vec{u}}{\partial t}$$

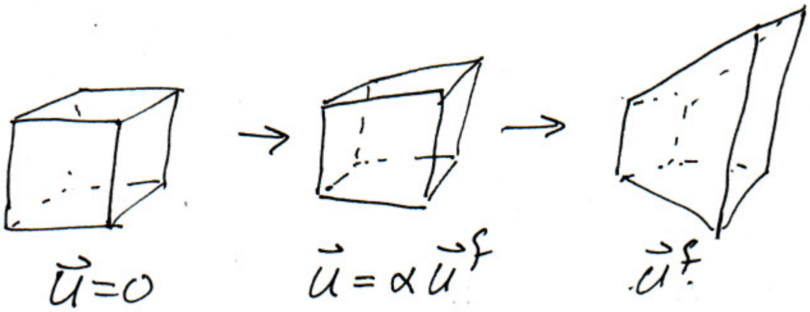
$$(\rho\, dV)\, \partial_0^2 u_i = \partial_j \sigma_{ij}\, dV$$

$$\rho\, \partial_0^2 u_i = \partial_j \sigma_{ij} = \partial_j \left(c_{ijkl}\, \varepsilon_{kl} \right)$$

[E&H notation: $\rho_0 \frac{\partial^2 \vec{\rho}}{\partial t^2} = \vec{\nabla} \cdot \underset{\text{strain tensor}}{\underset{\sim}{F}}$]

We'll return to this...

The Elastic Energy



Parameterize path from $\vec{u}=0$ to final $\vec{u}=\vec{u}^f$

linearly: $\vec{u}=\alpha\vec{u}^f$ $0<\alpha<1$.

Consider the work done by the body on the external world as $\vec{u}\to\vec{u}+\delta\vec{u} = \alpha\vec{u}^f+(\delta\alpha)\vec{u}^f$

Notice: this is minus the energy flow into the body.

The work done equals

$$\int\left(\frac{\text{Force}}{\text{volume}}\right)\cdot\delta\vec{u}\,dV = \int\underbrace{(\partial_j\sigma_{ij})}_{du}\cdot\underbrace{(\delta u_i)}_{v}\,dV$$

More fun with integration by parts!

$u=\sigma_{ij}$ $dv=\partial_j(\delta u_i)$

Assume $\sigma_{ij}\to 0$ far away, drop uv term

$$= -\int\sigma_{ij}\,\partial_j(\delta u_i)\,dV$$

$$= -\int\sigma_{ij}\,\partial_j u_i^f\,dV\,\delta\alpha.$$

$$= -\int\sigma_{ij}\left(\tfrac{1}{2}(\partial_j u_i^f+\partial_i u_j^f)\right)dV\,\delta\alpha$$

$$= -\int\sigma_{ij}\,\varepsilon_{ij}^f\,dV\,\delta\alpha$$

Now, $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = C_{ijkl}\left(\tfrac{1}{2}\left(\partial_i \underset{\sim}{u_j} + \partial_j \underset{\sim}{u_i}\right)\right)$ $\quad \alpha u_j^f \quad \alpha u_i^f$

$= \alpha\,\sigma_{ij}^f$ [Hooke's law again]

Minus Energy Flow into Body $\alpha \to \alpha + \delta\alpha$ $= -\int \sigma_{ij}^f \varepsilon_{ij}^f \, dV \;\; \alpha\,\delta\alpha$

Total Energy Stored in Body $= \int \sigma_{ij}^f \varepsilon_{ij}^f \, dV \; \underbrace{\int_0^1 \alpha\, d\alpha}_{1/2}$

$= \frac{1}{2}\int \sigma_{ij}\,\varepsilon_{ij}\, dV$

Energy Density in Elastic Medium $= \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij} = \frac{1}{2} C_{ijkl}\,\varepsilon_{ij}\,\varepsilon_{kl}$

If we treat this as the <u>definition</u> of C_{ijkl}, then we see that we can assume the tensor of elasticity has another symmetry

$$C_{ijkl} = C_{klij}$$

which Elmore & Heald call the <u>reciprocity relation</u>.

Isotropic Solids

You'll show (7.4.2) explicitly that there are only two independent elastic constants for isotropic materials. It turns out we can write C_{ijkl} in terms of the identity tensor (Kronecker delta) δ_{ij}:

$$C_{ijkl} = 2\mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$$

- Has all the symmetries $C_{ijkl} = C_{ijlk} = C_{jikl} = C_{klij}$

- Stress $\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} = (2\mu\,\underbrace{\delta_{ik}}_{\text{set } k=i}\overbrace{\delta_{jl}}^{\text{set } l=j} + \lambda\delta_{ij}\delta_{kl})\varepsilon_{kl}$

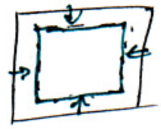
$$= 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} = \frac{Y}{1+\sigma}\varepsilon_{ij} + \frac{Y\sigma}{(1+\sigma)(1-2\sigma)}\varepsilon_{kk}\delta_{ij}$$

- Energy $= \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

$$= \mu\varepsilon_{ij}\varepsilon_{ij} + \frac{1}{2}\lambda(\varepsilon_{kk})^2$$

Relation to Other Elastic Constants

Bulk Compression $P = -K \frac{\Delta V}{V}$



$$\frac{\Delta V}{V} = \frac{(L+\Delta L)^3 - L^3}{L^3} \approx 3\frac{\Delta L}{L}$$

$$\varepsilon_{ij} = \frac{\Delta L}{L}\delta_{ij} = \begin{pmatrix} \frac{\Delta L}{L} & 0 & 0 \\ 0 & \Delta L/L & 0 \\ 0 & 0 & \Delta L/L \end{pmatrix}$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}$$

$$= 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\delta_{ij} = 2\mu\frac{\Delta L}{L}\delta_{ij} + \lambda\left(\frac{3\Delta L}{L}\right)\delta_{ij}$$

$$= (2\mu + 3\lambda)\frac{\Delta L}{L}\delta_{ij} = \left(\frac{2\mu}{3} + \lambda\right)\frac{\Delta V}{V}\delta_{ij}$$

Hydrostatic Pressure: $\sigma_{ij} = -P\,\delta_{ij}$

$$P = -\left(\frac{2\mu}{3} + \lambda\right)\frac{\Delta V}{V}$$

$$\boxed{K = \frac{2\mu}{3} + \lambda}$$

Shear:

$\partial_y u_x = \theta$

Is Energy $= \frac{1}{2}\mu\theta^2$?

$$\varepsilon_{ij} = \begin{pmatrix} 0 & \theta/2 & 0 \\ \theta/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

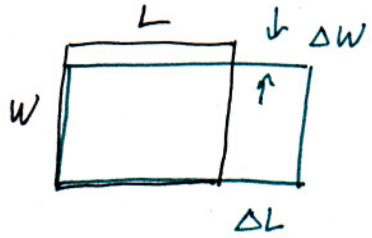
$$E = \mu\,\varepsilon_{ij}^2 + \frac{\lambda}{2}\varepsilon_{ii}^2 = \mu\sum_i\sum_j \varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}\Big(\sum_i \varepsilon_{ii}\Big)\Big(\sum_j \varepsilon_{jj}\Big)$$

$$= \mu\left(\left(\frac{\theta}{2}\right)^2 + \left(\frac{\theta}{2}\right)^2\right)$$

$$= \frac{1}{2}\mu\theta^2$$

$\boxed{\text{Shear Modulus} = \mu}$

Stretching (Unconstrained)



Needed:

- Stretch $\Delta L / L = a$
- Shrink $b = \frac{\Delta W}{W} = \frac{\Delta H}{H}$

to minimize energy density

- $b = \sigma a$
- $E = \frac{1}{2} Y a^2$

$$\varepsilon_{ij} = \begin{pmatrix} \Delta L/L & 0 & 0 \\ 0 & -\frac{\Delta W}{W} & 0 \\ 0 & 0 & -\frac{\Delta W}{W} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -b \end{pmatrix}$$

$$\text{Energy density} = \mu \varepsilon_{ij}^2 + \frac{\lambda}{2} \varepsilon_{ii}^2$$

$$= \mu \sum_i \sum_j \varepsilon_{ij} \varepsilon_{ij} + \frac{\lambda}{2} \left(\sum_i \varepsilon_{ii} \right) \left(\sum_j \varepsilon_{jj} \right)$$

$$= \mu \left(a^2 + 2b^2 \right) + \frac{\lambda}{2} \left(a - 2b \right)^2$$

$$= \left(\mu + \frac{\lambda}{2} \right) a^2 + \left(2\mu + 2\lambda \right) b^2 - 2\lambda ab$$

$$\frac{\partial \text{Energy}}{\partial b} = 4(\mu + \lambda) b - 2\lambda a = 0 \Rightarrow b = \frac{2\lambda}{4(\mu+\lambda)} a$$

$$\boxed{\sigma = \frac{\lambda}{2(\mu+\lambda)}}$$

$$\ldots E = \frac{1}{2} \left(\frac{2\mu^2 + 3\lambda\mu}{\mu + \lambda} \right) a^2 \qquad \boxed{Y = \frac{2\mu^2 + 3\lambda\mu}{\mu + \lambda}}$$