The Wave Equation for Isotropic Solids

$$ma = F$$

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \partial_j \sigma_{ij}$$

$$= \partial_j c_{ijkl} \varepsilon_{kl}$$

$$= \partial_j c_{ijkl} \left(\frac{\partial_k u_l + \partial_l u_k}{2} \right)$$

Ignore geometric nonlinearity

$$c_{ijkl} = \cancel{2\mu \delta_{ik} \delta_{jl}} + \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \mu \delta_{ik} \delta_{jl} \left(\partial_j \partial_k u_l + \partial_j \partial_l u_k \right) + \frac{\lambda}{2} \delta_{ij} \delta_{kl} \left(\partial_j \partial_k u_l + \partial_j \partial_l u_k \right)$$

$$= \mu \left(\partial_j \partial_i u_j + \partial_j \partial_j u_i \right) + \frac{\lambda}{2} \left(\partial_i \partial_k u_k + \partial_i \partial_k u_k \right)$$

$$= (\mu + \lambda) \partial_i \partial_j u_j + \mu \partial_j \partial_j u_i$$

$$\boxed{\rho_0 \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu) \vec{\nabla} (\vec{\nabla} \cdot \vec{u}) + \mu \nabla^2 \vec{u}}$$

Can we find our old wave function inside this one in disguise?

Helmholtz' Theorem: Any vector field $\vec{u}$ can be written as the sum

$$\vec{u} = \vec{u}_L + \vec{u}_T$$

with $\vec{\nabla}\times\vec{u}_L = 0$ and $\vec{\nabla}\cdot\vec{u}_T = 0$.

Messy! not true on a torus!

Proof: [Math class?]

$$u_T = \nabla\times\left(\frac{1}{4\pi}\int\frac{\nabla\times u(x')}{|\vec{x}-\vec{x}'|}d^3x'\right) \quad \vec{u}_L = -\nabla\left(\frac{1}{4\pi}\int\frac{\nabla\cdot u(x')}{|x-x'|}d^3x'\right)$$

$$= \nabla\left(\frac{1}{4\pi}\int\frac{\nabla\cdot u}{|x-x'|}d^3x'\right)$$

$$- \frac{1}{4\pi}\int\frac{\nabla^2 u}{|x-x'|}d^3x'$$

$$\vec{u}_T + \vec{u}_L = -\frac{1}{4\pi}\int\frac{\nabla^2\vec{u}}{|\vec{x}-\vec{x}'|}d^3x' = \vec{u}$$

Know from Coulomb's Law $u_x = \phi$

$$\vec{\nabla}\phi = \vec{E}$$

$$\mathrm{div}\,\vec{E} = 4\pi\rho = \nabla^2\phi$$

$$-\frac{1}{4\pi}\int\frac{\nabla^2\phi}{|x-x'|}d^3x' = -\frac{1}{4\pi}\int\frac{4\pi\rho}{|x-x'|}d^3x'$$

$$= -\sum\frac{e^2}{r} = \phi \quad \checkmark$$

Transverse Sound

- A transverse plane wave has no divergence

$$\vec{u}_T(x) = \hat{y}\, e^{i(kx-\omega t)}$$

$$\vec{\nabla}\cdot\vec{u}_T = \cancel{\partial_x u_x} + \cancel{\partial_y} u_y + \cancel{\partial_z u_z} = 0$$

$$\rho_0 \frac{\partial^2 u_T}{\partial t^2} = (\lambda+\mu)\cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{u}_T)} + \mu \nabla^2 u$$

$$\boxed{\frac{\partial^2 u_i}{\partial t^2} = \frac{\mu}{\rho_0}\partial_j^2 u_i = \frac{\mu}{\rho_0}\vec{\nabla}^2 u_i}$$

- Three 3-dimensional wave equations

(Zero divergence $\Rightarrow$ really only two waves for each $\vec{k}$: For $\vec{k} \sim k\hat{x}$, wave polarized along $\hat{y}$ and along $\hat{z}$).

- Transverse speed of sound $c_t = \sqrt{\frac{\mu}{\rho}}$

- $c_t = 0$ for fluids (since shear modulus $\mu = 0$)

Longitudinal Sound

- A longitudinal plane wave has no curl

$$\vec{u}_L(x) = \hat{x}\, e^{i(kx-\omega t)}$$

$$\vec{\nabla}\times\vec{u}_L = (\cancel{\partial_y u_z} - \cancel{\partial_z u_y},\ \cancel{\partial_z u_x} - \cancel{\partial_x u_z},\ \partial_x \cancel{u_y} - \cancel{\partial_y u_x})$$

- $(\vec{\nabla}\times\overset{0}{\overset{\|}{\vec{\nabla}\times\vec{u}_L}})_i = \epsilon_{ijk}\,\partial_j\,\epsilon_{k\ell m}\partial_\ell u_m$

$$= \underbrace{\epsilon_{ijk}\,\epsilon_{k\ell m}}\,\partial_j\partial_\ell u_m$$

cyclic: $\epsilon_{k\ell m} = \epsilon_{\ell m k}$

$$\epsilon_{ijk}\,\epsilon_{\ell m k} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

$i=\ell$ or $i=m$

$\Downarrow \qquad \Downarrow$

$j=m \qquad j=\ell$

$\Downarrow \qquad \Downarrow$

$1 \qquad -1$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\,\partial_j\partial_\ell u_m$$

$$= \partial_m\partial_i u_m - \partial_j\partial_j u_i = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \nabla^2\vec{u}$$

Can swap if $\nabla\times u_L = 0$!

- $\rho_0 \dfrac{\partial^2\vec{u}_L}{\partial t^2} = (\lambda+\mu)\underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{u}_L)}_{\nabla^2\vec{u}_L} + \mu\,\vec{\nabla}^2 u_L$

$$\rho_0\frac{\partial^2\vec{u}_L}{\partial t^2} = (\lambda+2\mu)\,\nabla^2\vec{u}_L$$

- Again, three 3-D wave equations (Zero curl → really only one wave for each $\vec{k}$, polarized parallel to $\vec{k}$).

- Longitudinal sound speed (thick wires, bulk solids) $c_\ell = \sqrt{\dfrac{\lambda + 2\mu}{\rho}}$