TRAVELING WAVES

Let's guess a family of solutions to the wave equation.

Clues:

<u>Sound</u> Travels undistorted from mouth to ear
(Not quite: close lightning CRACK! far RRUMBLE...)

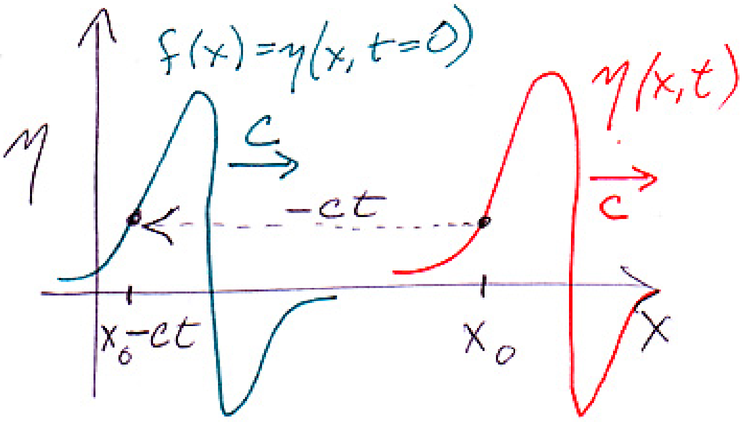
<u>Pulse on Spring</u> Any Shape
(A bit crummy, too fast). Try <u>longitudinal</u>.

<u>Pulse on Torsional Wave Machine</u> Any Shape.

- If sound is undistorted, then wave must propagate without changing shape.

- If pulses retain shape, then all pulses must move at the <u>same</u> velocity.

What velocity? $c = \sqrt{\tau/\lambda_0}$!



Exercise:
What is $\eta(x,t)$, in terms of shape $f(x)$ at $t=0$, velocity c, and time t?

If pulse moves right, need to shift x_0 by $-ct$ to find out where it was.

Rightward-moving pulse
$$\eta(x,t) = f(x-ct)$$
[Leftward $\eta(x,t) = f(x+ct)$]

Does $\eta(x,t) = f(x-ct)$ satisfy the wave equation? Write $f'(z) = df/dz$

$$\frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}\left(f(x-ct)\right) = f'(x-ct)\frac{\partial}{\partial t}(x-ct) = -c\, f'(x-ct)$$

$$\frac{\partial \eta}{\partial x} = f'(x-ct)$$

$$\frac{\partial^2 \eta}{\partial t^2} = -c\frac{\partial}{\partial t} f'(x-ct) = c^2 f''(x-ct)$$

$$\frac{\partial^2 \eta}{\partial x^2} = f''(x-ct)$$

$$\frac{\partial^2 \eta}{\partial t^2} = c^2 f''(x-ct) = c^2 \frac{\partial^2 \eta}{\partial x^2} \quad \checkmark$$

$\boxed{\text{Traveling wave } \eta(x,t) = f(x-ct) \text{ satisfies the wave equation for any wave shape } f!}$

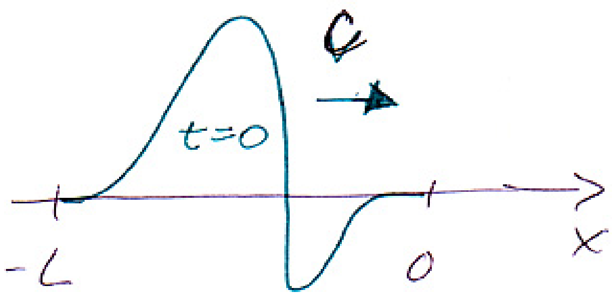
$c = \tau/x_0$ = Pulse velocity = velocity of sound

Exercise: Are these traveling waves? What is c if they are?

$$\eta(x,t) = \frac{2}{(x-4t)^2+1} \qquad \eta(x,t) = e^{-[(7x+8t)^2+37]}$$

$$\eta(x,t) = x^2+t^2 \qquad \eta(x,t) = \cos(kx-\omega t)$$

Exercise:

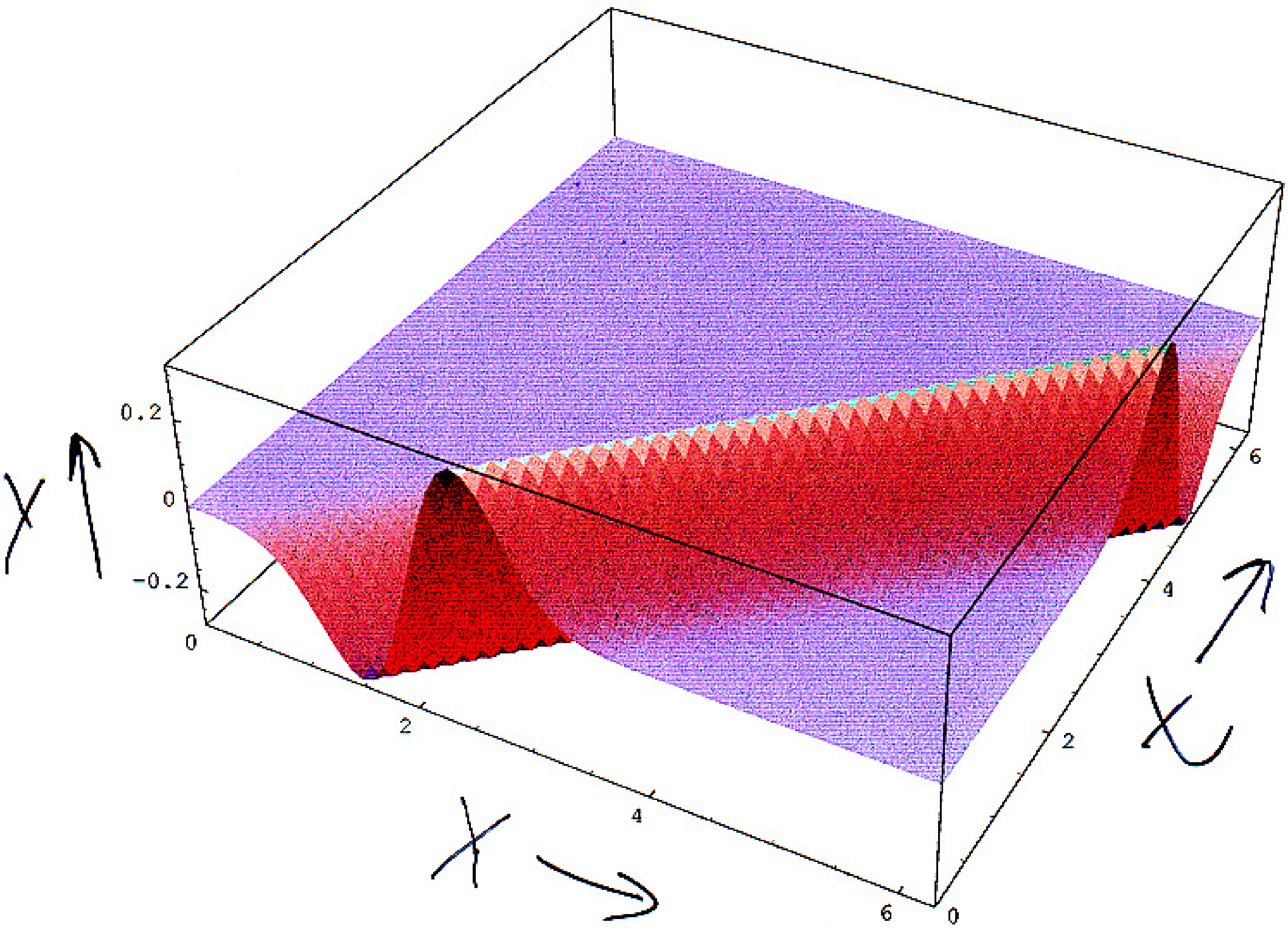


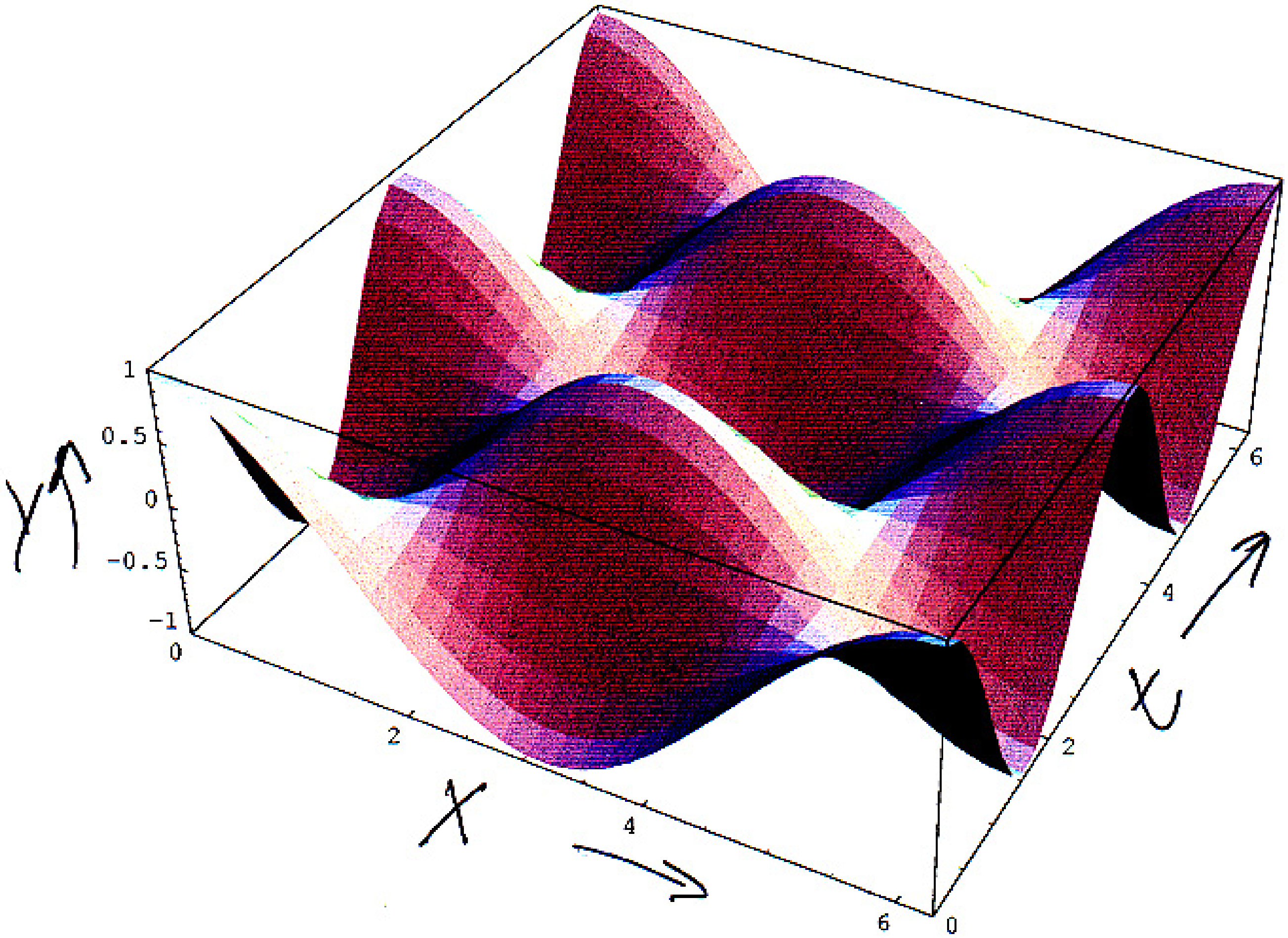
Pulse with funny shape at $t=0$ travels to right at velocity $c = \sqrt{T/\mu_0}$.

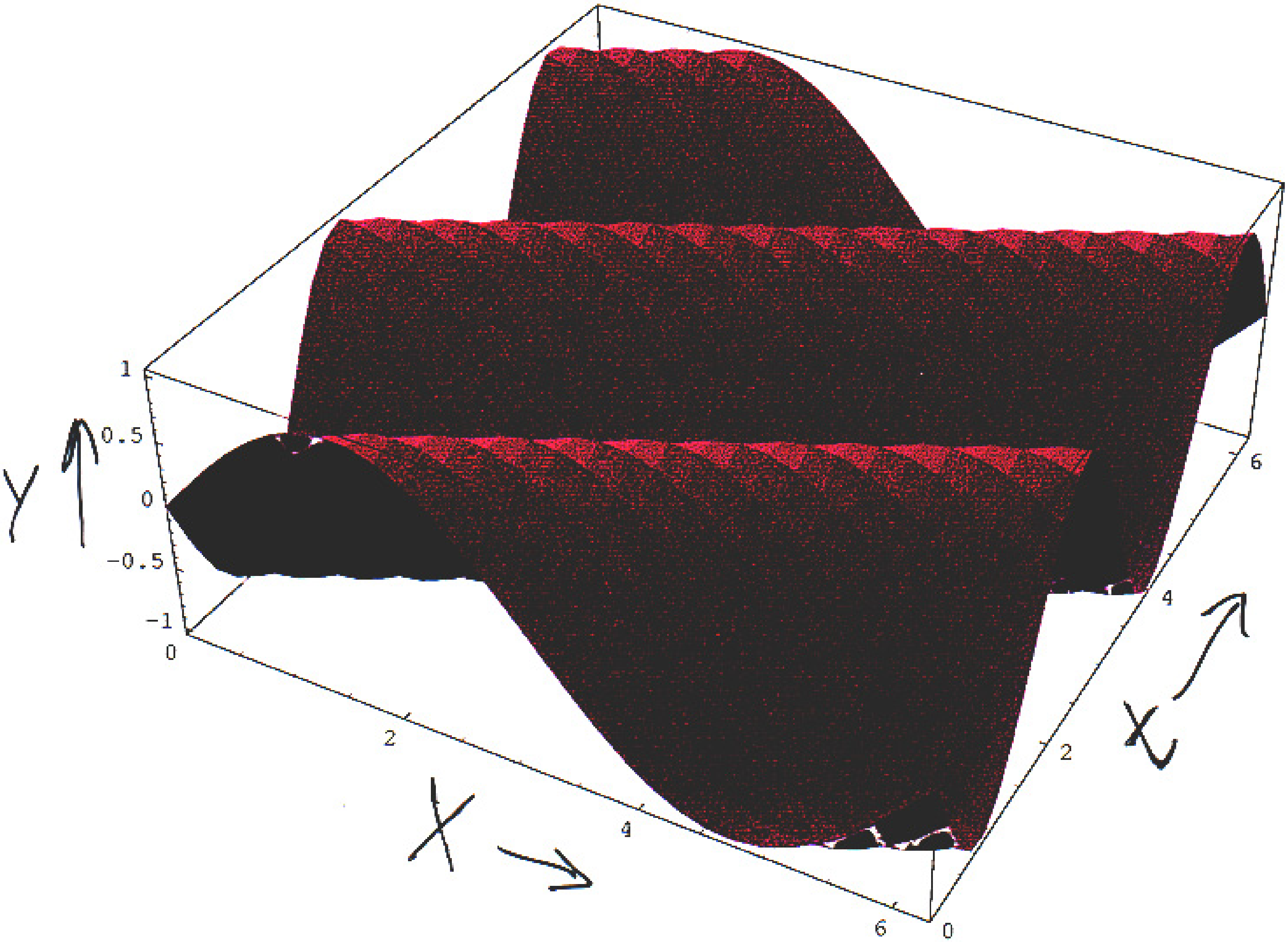
Draw $\eta(0,t)$, the height of the chunk of string at $x=0$.

$\eta(0,t) = f(-ct)$

Exercise: Is this a traveling wave?


0.2
0
-0.2
0
2
4
6
0
2
4
6
X
t
X

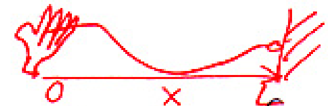

1
0.5
0
-0.5
-1
0
2
4
6
0
2
4
6
Y
X
t


1
0.5
0
-0.5
-1
0
2
4
6
0
2
4
6
Y
X
t

Demo: Pulse on Spring:
Inverts At Wall

Demo: Pulse on Torsional Wave Machine
Does Not Invert

Spring

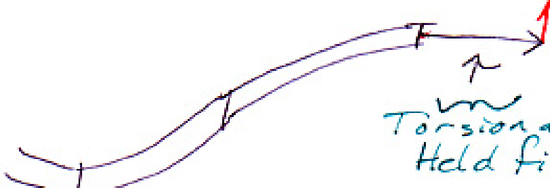


Boundary Conditions at the Ends of a String

Fixed boundary condition at wall $\quad y(L,t) \equiv 0$

Fixed, forced boundary condition at hand $\quad y(0,t) = f(t)$

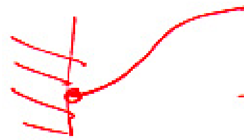
Free boundary condition at end of wave machine $\quad \frac{\partial y}{\partial x}(L,t) = 0$



$\tau \tan\theta = \tau \frac{\partial y}{\partial x} = 0$ } No force in y-direction

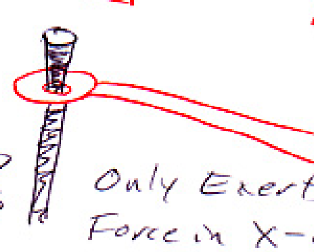
Torsional Rod
Held fixed in
x-direction

Fixed Boundary



$\frac{\partial y}{\partial t} = 0 \Rightarrow y(x_0,t) = \text{cons}$

Free BC



$\frac{\partial y}{\partial x} = 0$

Frictionless Rod:

Only Exerts Force in x-direction

Superposition

Suppose $y_1(x,t)$ and $y_2(x,t)$ satisfy the wave equation. Then

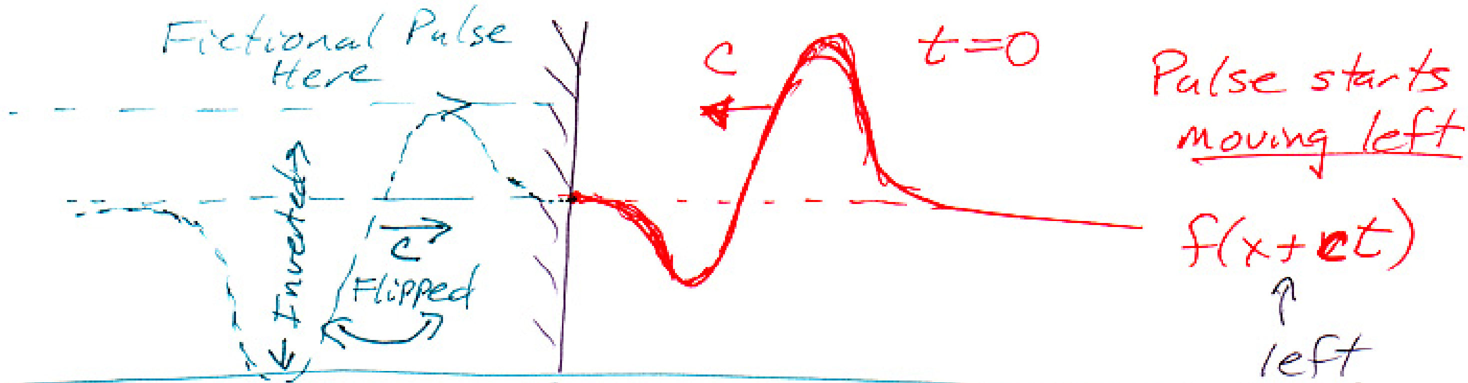
$$\frac{\partial^2(y_1+y_2)}{\partial t^2} = \frac{\partial^2 y_1}{\partial t^2} + \frac{\partial^2 y_2}{\partial t^2}$$

$$= c^2\frac{\partial^2 y_1}{\partial x^2} + c^2\frac{\partial^2 y_2}{\partial x^2} = c^2\frac{\partial^2}{\partial x^2}(y_1+y_2)$$

so $y_1(x,t)+y_2(x,t)$ <u>also</u> solves the wave equation.
True for all <u>linear</u> equations

Fixed Boundaries & Pulses

Consider a pulse from the <u>left</u> impinging on a wall at $x=0$:



Exercise: Draw a fictional pulse, moving from behind the wall to the right, which when added ~~to~~ (superimposed) will guarantee that $y(0,t)=0$.

Pulse flips and inverts at fixed boundary; proof by superposition

Exercise: What is the equation for the flipped, inverted pulse?

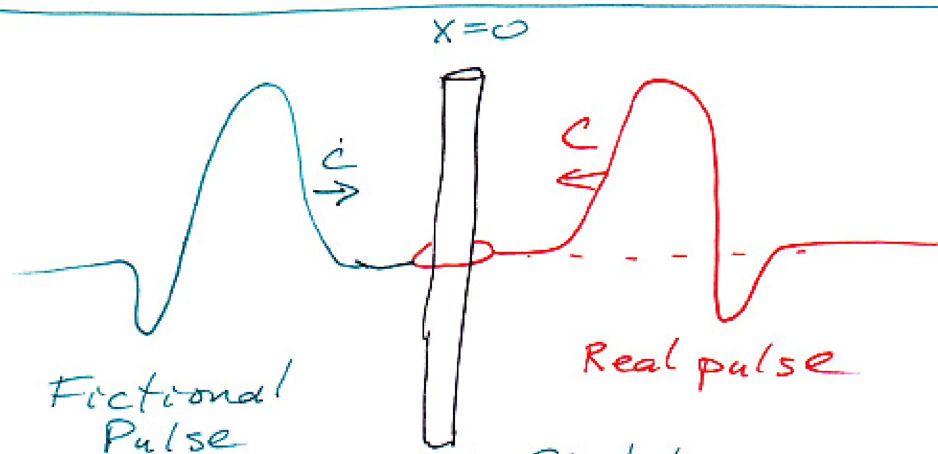
$-f((x-ct))$
flipped
inverted
right

Exercise: Show algebraically that the boundary condition is satisfied.

$$y(x,t) = f(x+ct) - f(-(x-ct))$$

$$y(0,t) = f(ct) - f(-(-ct)) = f(ct) - f(ct) = 0$$

Exercise: What fictional pulse for $x<0$ would enforce free boundary conditions?



$$f(x+ct) + f(-(x-ct))$$

Right

not inverted

flipped

Proof by superposition! pulse flips, does not invert at free boundary