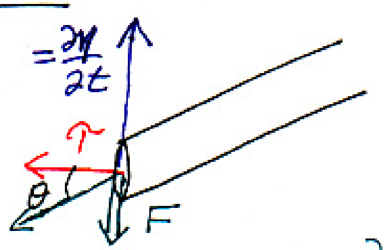
Energy and Power for Waves on Strings

Applications:

Microwaves Sound (dB)
Sunlight Transmission Lines

Conservation Laws

Power



Exercise: Is the power positive (energy flow to the right) or negative (left)?

$F = -\tau \tan\theta = -\tau \frac{\partial \eta}{\partial x}$

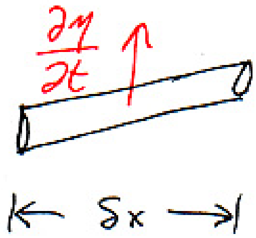
$$\boxed{\text{Power} = F \cdot V = -\tau \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial t}}$$

Demo: Wave on a Spring

Exercise: Estimate τ $\sim 1-10\ N$
Estimate $\frac{\partial \eta}{\partial x}$ $\sim 0.1-0.3$ rad
Estimate $\partial\eta/\partial t$ ~ 0.1 m/s

Power $\sim 0.01 - 0.3$ W (Electrical: bigger pipe)

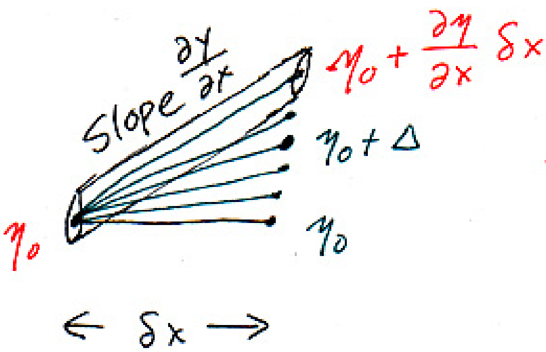
Energy



Chunk Kinetic Energy $= \frac{1}{2} m v^2$

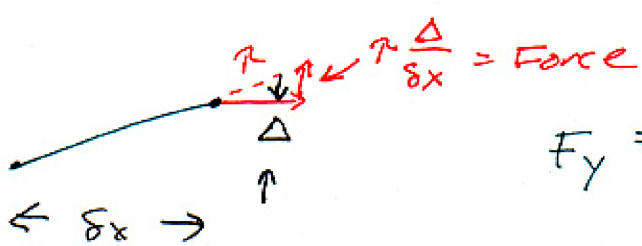
$$= \underbrace{\frac{1}{2} \lambda_0 \left(\frac{\partial \eta}{\partial t}\right)^2}_{\text{KE Density } K} \delta x$$

$$\text{Total KE} = \int \frac{1}{2} \lambda_0 \left(\frac{\partial \eta}{\partial t}\right)^2 dx$$



Chunk Potential Energy

$$= \int_{0 \,(\text{zero slope})}^{\frac{\partial \eta}{\partial x}\delta x \,(\text{final slope})} \underbrace{F_y}_{\text{Force}} \cdot \underbrace{d\Delta}_{\text{Distance}}$$



$$F_y = \tau \frac{\Delta}{\delta x}$$

Chunk Potential Energy

$$= \int_0^{\left(\frac{\partial \eta}{\partial x}\delta x\right)} \tau \frac{\Delta}{\delta x} d\Delta = \frac{\tau}{\delta x}\left(\frac{\Delta^2}{2}\right)\Bigg|_0^{\frac{\partial \eta}{\partial x}\delta x}$$

$$= \frac{1}{2} \tau \left(\frac{\partial \eta}{\partial x}\right)^2 \delta x$$

$$\text{Total Potential Energy} = \int \underbrace{\frac{1}{2} \tau \left(\frac{\partial \eta}{\partial x}\right)^2}_{\text{PE Density } V} dx$$

$$\text{Total Energy} = \int \underbrace{\left[\frac{1}{2} \lambda_0 \left(\frac{\partial \eta}{\partial t}\right)^2 + \frac{1}{2} \tau \left(\frac{\partial \eta}{\partial x}\right)^2\right]}_{\text{Energy Density } E = K + V} dx$$

Is energy conserved?

$$\frac{dE}{dt} = \frac{d}{dt}\int_0^L \tfrac{1}{2}\lambda_0\left(\frac{\partial \eta}{\partial t}\right)^2 + \tfrac{1}{2}\tau\left(\frac{\partial \eta}{\partial x}\right)^2 dx$$

Integrate by Parts

$$u = \tau\frac{\partial \eta}{\partial x} \quad dv = \frac{\partial^2 \eta}{\partial x \partial t}$$

$$du = \tau\frac{\partial^2 \eta}{\partial x^2} \quad v = \frac{\partial \eta}{\partial t}$$

$$= \int_0^L \lambda_0 \frac{\partial \eta}{\partial t}\frac{\partial^2 \eta}{\partial t^2}\,dx + \underbrace{\int_0^L \tau\frac{\partial \eta}{\partial x}\frac{\partial^2 \eta}{\partial t \partial x}\,dx}_{uv|_0^L - v\,du}$$

$$\tau\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}\Big|_0^L - \int_0^L \tau\frac{\partial \eta}{\partial t}\frac{\partial^2 \eta}{\partial x^2}\,dx$$

$$= \underbrace{\tau\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}\Big|_L}_{\text{Power flow to left from "wall"}} \underbrace{-\,\tau\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}\Big|_0}_{\text{Power flow to right from "hand"}} + \int_0^L \frac{\partial \eta}{\partial t}\underbrace{\cancel{\left(\lambda_0\frac{\partial^2 \eta}{\partial t^2} - \tau\frac{\partial^2 \eta}{\partial x^2}\right)}}_{\text{Wave Equation} = 0}dx$$

<u>Yes</u>.

Exercise: What is the energy flow through a fixed boundary? A free boundary?

$$P = -\tau\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}$$

Zero for fixed boundary

Zero for free boundary

<u>Zero</u>.

Conservation of Momentum

What is the momentum g_x of a wave $y(x,t)$ in the x-direction?

- Our derivation suggests it is <u>zero</u>: all motion is transverse.

- General mumbo-jumbo (learn in your next Mechanics class) says energy-conserving "Hamiltonian" systems with <u>homogeneity</u> always have a conserved <u>momentum</u>.

  <u>Time independence</u> in Hamiltonian systems implies energy conservation. To every continuous symmetry, they will show, corresponds a conservation law.

- Elmore & Heald, ch. 1.10 & 1.11 do better wave equation, derive conservation of momentum.

  The final formula for the momentum makes no reference to longitudinal motion of the wire!

Can we derive it? [Long, formal calculation]

To show energy conservation, the last step involved the wave equation $\lambda_0 \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2} = 0$ multiplied by $\partial y/\partial t$.

What happens if we multiply by $\partial y/\partial x$, and work backwards?

$$0 = \frac{\partial y}{\partial x}\left(\lambda_0 \frac{\partial^2 y}{\partial t^2} - \tau \frac{\partial^2 y}{\partial x^2}\right)$$

$$= \underbrace{\lambda_0 \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial t^2}}_{\lambda_0\left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right) - \underbrace{\frac{\partial^2 y}{\partial t \partial x}\frac{\partial y}{\partial t}}_{\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial t}\right)^2}\right]} \underbrace{- \tau \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x^2}}_{-\frac{1}{2}\tau\frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)^2}$$

$$= \frac{\partial\left(\lambda_0 \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right)}{\partial t} - \frac{\partial}{\partial x}\Big(\underbrace{\tfrac{1}{2}\lambda_0\left(\frac{\partial y}{\partial t}\right)^2}_{K}\Big) - \frac{\partial}{\partial x}\Big(\underbrace{\tfrac{1}{2}\tau\left(\frac{\partial y}{\partial x}\right)^2}_{V}\Big)$$

$$= -\frac{\partial g_x}{\partial t} - \frac{\partial E}{\partial x}$$

where $g_x = -\lambda_0 \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$ is the momentum density. If $G_x = \int g_x \, dx$ is the total momentum,

$$\frac{dG_x}{dt} = \int \frac{\partial g_x}{\partial t} dx = \int -\frac{\partial E}{\partial x} dx = -E\Big|_0^L$$

so E = momentum flow to the right.

Energy Density

Notice: $P = -\tau \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} = \frac{\tau}{\lambda_0} g_x = c^2 g_x$
true for all wave equations.