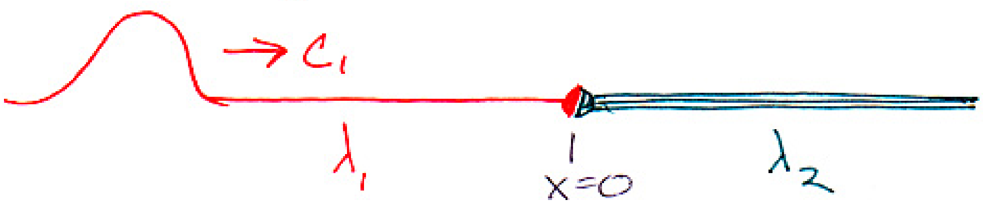
REFLECTION AND TRANSMISSION

Demo: Two torsional wave machines

Two different strings, λ_1 and λ_2 mass/length, tied together with massless knot at $x=0$.



Initial pulse $A_1 f(\underbrace{t-\frac{x}{c_1}}_{\text{OK!}})$ heading toward knot.

What will the reflected and transmitted pulses look like?

$f(t)$ = Motion of knot due to incident pulse, if there were no discontinuity.

Can we guess a solution? Suppose the reflected pulse is of the same shape (flipped, perhaps inverted): $B_1 f(t+\frac{x}{c})$

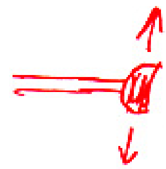
Inverted if $B_1 < 0$ ↗ ↑ Flipped

Now the net motion of the red string is

$$y_1(x,t) = A_1 f\left(t - \frac{x}{c_1}\right) + B_1 f\left(t + \frac{x}{c_1}\right)$$

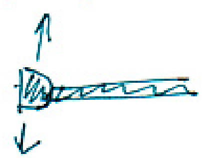
and the left-hand side of the knot moves as

$$(A_1 + B_1) f(t)$$



Clearly, the transmitted wave on string 2 must have the same motion at $x=0$, so its shape in time must also be a constant times $f(t)$

$$y_2(x,t) = A_2 f\left(t - \frac{x}{c_2}\right)$$



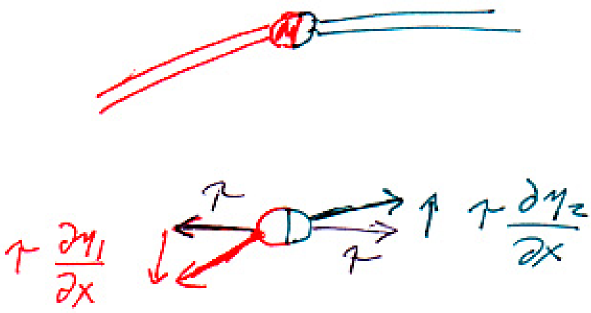
$$A_2 f(t) = (A_1 + B_1) f(t)$$

with $\boxed{A_2 = A_1 + B_1}$ (Continuity of height $y(x)$).

We'd like to know:

(1) Does this obey Newton's law at the knot?

(2) What are A_2 and B_1, the transmitted and reflected amplitudes?



Knot free-body diagram

$$ma = \tau\left(\frac{\partial y_2}{\partial x} - \frac{\partial y_1}{\partial x}\right)$$

Massless knot
$\Rightarrow$ slopes must match!

$$\frac{\partial y_1}{\partial x} = -\frac{A_1}{c_1} f'\left(t - \frac{x}{c_1}\right) + \frac{B_1}{c_1} f'\left(t + \frac{x}{c_1}\right) = \frac{B_1 - A_1}{c_1} f'(t)$$

At knot

$$\frac{\partial y_2}{\partial x} = -\frac{A_2}{c_2} f'\left(t - \frac{x}{c_2}\right) = -\frac{A_2}{c_2} f'(t)$$

At knot

At knot $x = 0$, these must be equal

$$\boxed{\frac{B_1 - A_1}{c_1} = -\frac{A_2}{c_2}}$$

Continuity of Slope at Knot

Let $R = B_1/A_1$ be the <u>reflection coefficient</u>
$T = A_2/A_1$ be the <u>transmission coefficient</u>

$$\frac{A_2}{A_1} = \frac{A_1 + B_1}{A_1} \Rightarrow \boxed{T = 1 + R}$$

Continuity of String

$$\frac{1}{c_1}\left(\frac{B_1 - A_1}{A_1}\right) = -\frac{1}{c_2}\frac{A_2}{A_1} \Rightarrow \boxed{\frac{1}{c_1}(R - 1) = -\frac{1}{c_2} T}$$

Continuity of Slope

These can be solved for R and T:

$$R = T - 1$$

$$\frac{1}{c_1}(T-2) = -\frac{1}{c_2}T$$

$$\left(\frac{1}{c_1}+\frac{1}{c_2}\right)T = \frac{c_1+c_2}{c_1c_2}T = \frac{2}{c_1}.$$

$$T = \frac{2c_2}{c_1+c_2}$$

$$R = T - 1 = \frac{c_2-c_1}{c_1+c_2}$$

showing that this solves Newton's laws at the knot (and hence is the correct solution).

Exercise: What are R and T in terms of λ_1 and λ_2?

$$c = \sqrt{\tau/\lambda} \qquad T = \frac{2\sqrt{\tau/\lambda_2}}{\sqrt{\tau/\lambda_1}+\sqrt{\tau/\lambda_2}}\left(\frac{\sqrt{\lambda_1\lambda_2/\tau}}{\sqrt{\lambda_1\lambda_2/\tau}}\right) = \frac{2\sqrt{\lambda_1}}{\sqrt{\lambda_2}+\sqrt{\lambda_1}}$$

$$R = \frac{\sqrt{\lambda_1}-\sqrt{\lambda_2}}{\sqrt{\lambda_2}+\sqrt{\lambda_1}}$$