BEATS, Group & Phase Velocities, AM & FM

Mathematical Truth:

$$e^{iA} + e^{iB} = e^{i\left(\frac{A+B}{2}\right)}\left(e^{i\left(\frac{A-B}{2}\right)} + e^{i\left(\frac{B-A}{2}\right)}\right)$$

$$= e^{i\left(\frac{A+B}{2}\right)}\, 2\cos((A-B)/2)$$

Consequences:

(1) Angle addition formulas: Real and Imaginary Parts

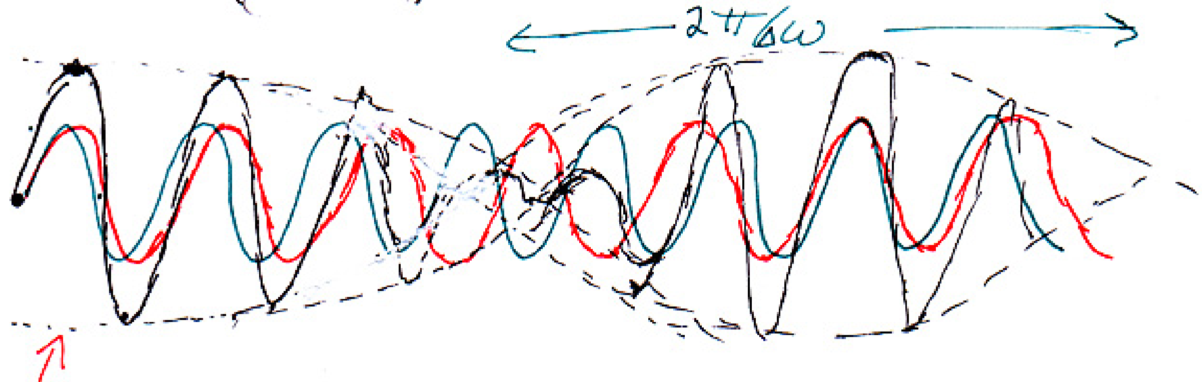
$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

(2) Beats: $A = (\omega + \frac{\Delta\omega}{2})t$, $B = (\omega - \frac{\Delta\omega}{2})t$, $\Delta\omega$ small

DEMO

$$\cos\left(\left(\omega + \frac{\Delta\omega}{2}\right)t\right) + \cos\left(\left(\omega - \frac{\Delta\omega}{2}\right)t\right) = 2\cos(\omega t)\cos\left(\frac{\Delta\omega}{2}t\right)$$

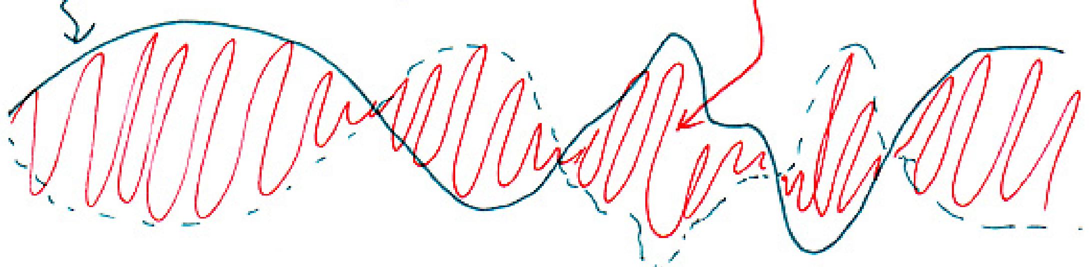


Envelope $= \pm 2\cos\left(\frac{\Delta\omega t}{2}\right)$ (Smooth curve over wiggles)

Two beats per period: period $= \frac{4\pi}{\Delta\omega}$, beat every $\frac{2\pi}{\Delta\omega}$

(3) AM Radio

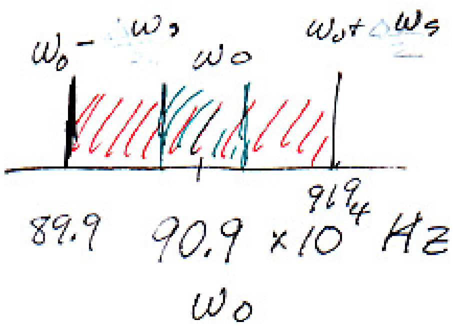
Amplitude Modulation: microphone gives pressure $A(t)$ envelope on top of carrier wave $\cos\omega_0 t$



Bandwidth: Tuning fork at $f = \omega/2\pi = 20{,}000$ Hz

$$A(t) = A_0 \cos\omega_s t \cos\omega_0 t$$

$$= \frac{A_0}{2}\cos(\omega_0 + \omega_s) + \frac{A_0}{2}\cos(\omega_0 - \omega_s)$$

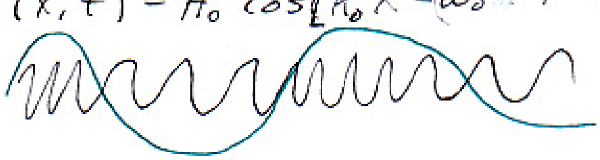


Band of 20000 Hz needed to carry full audible frequency range

Bandwidth ⇔ Information per unit time

- AM radio < 5 KHz
- Book has different envelope  ⇒ half the info.
- FM = Frequency Modulation

$$\psi(x,t) = A_0 \cos[k_0 x - (\omega_0 + A(t))t]$$



(4) Group & Phase Velocity

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$A = \left(k + \frac{\Delta k}{2}\right)x - \left(\omega + \frac{\Delta\omega}{2}\right)t$$

$$B = \left(k - \frac{\Delta k}{2}\right)x - \left(\omega - \frac{\Delta\omega}{2}\right)t$$

$$\cos\left(\left(k + \frac{\Delta k}{2}\right)x - \left(\omega + \frac{\Delta\omega}{2}\right)t\right) + \cos\left(\left(k - \frac{\Delta k}{2}\right)x - \left(\omega - \frac{\Delta\omega}{2}\right)t\right)$$

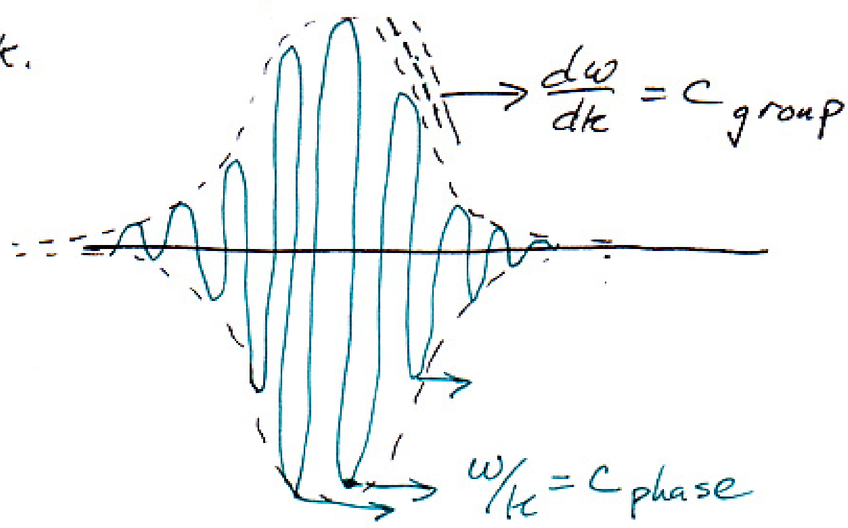
$$= 2\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos(kx - \omega t)$$

Envelope packet — Group velocity $\frac{\Delta\omega}{\Delta k}$

Carrier wave — Phase velocity ω/k

One-dimensional atom chain, etc: $\omega(k)$ can vary from ck.

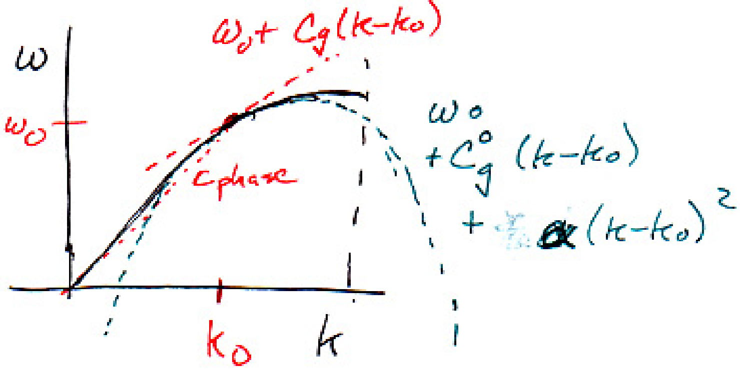
A slowly modulated wave of frequency ω will have wave crests which move at $c_{phase} = \omega/k$, but the signal (envelope) will move at $c_{group} = d\omega/dk$.



(5) Why it's called "dispersion"

Webster's: dis-perse \dis-'pərs\ ...
1a: to cause to break up ... b: to cause to become spread widely

How will the envelope of our packet change shape as it moves?

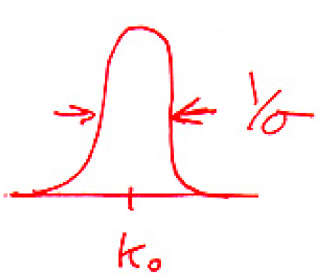


$c_g(k) \approx c_g^0 + 2\alpha(k-k_0)$
If $\alpha \neq 0$, group velocity slows down for larger k: peak will spread out.

How big a range of k does a packet of original width σ ($= \sqrt{2}\,\Delta x_0$ E&H 12.5) have?

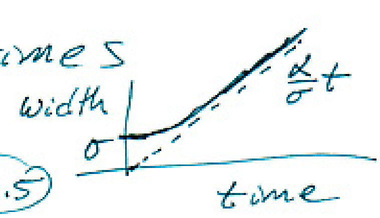
Pick Gaussian envelope at $t=0$ $\quad f(x) = Ae^{-x^2/2\sigma^2} e^{ik_0 x}$

~~Computer~~ Fourier Lab $\quad \tilde{f}(k) \propto e^{-\sigma^2(k-k_0)^2/2}$

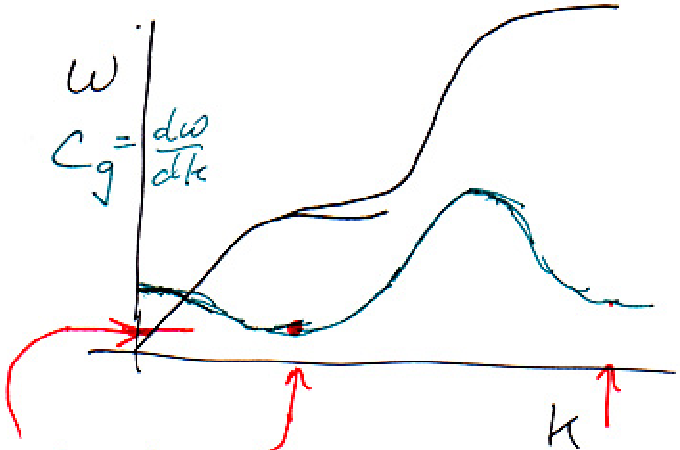


Range of k's $\sim 1/\sigma$
Range of velocities $\sim 2\alpha\,\Delta k \sim \alpha/\sigma$

Envelope broadens $\sim (\alpha/\sigma)t$ at late times
(agrees with E&H detailed calculation) (12.5)



Optical fibers designed so that laser frequency is at a local extremum of the group velocity



Packets smearing out along fiber degrades signal...

Good laser frequency! minimizes packet spread