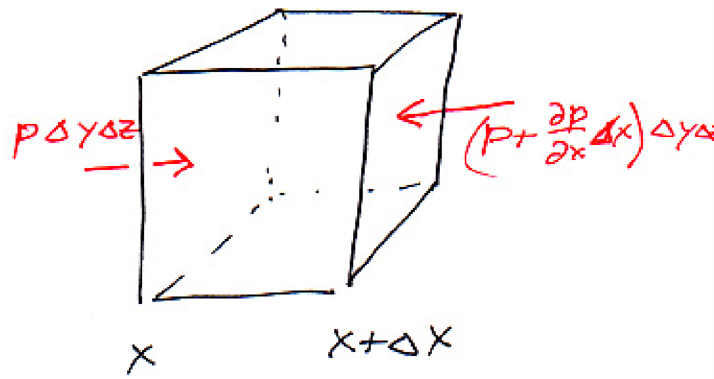
Three Dimensional Wave Equation for Fluids



Consider a cube of air of size $\Delta x, \Delta y, \Delta z$, pressure change $p(\vec{x}) = P(\vec{x}) - P_{atm}$

The force on the left is $p\,\Delta y\,\Delta z$; on the right is $\left(p + \frac{\partial p}{\partial x}\Delta x\right)\Delta y\,\Delta z$.

$$\left.\begin{aligned} F_x &= -\frac{\partial p}{\partial x}\,\Delta x\,\Delta y\,\Delta z \\ F_y &= -\frac{\partial p}{\partial y}\,\Delta x\,\Delta y\,\Delta z \\ F_z &= -\frac{\partial p}{\partial z}\,\Delta x\,\Delta y\,\Delta z \end{aligned}\right\} \quad \vec{F} = -\vec{\nabla} p\,\Delta x\,\Delta y\,\Delta z$$

Elmore & Heald call the fluid density ρ_0 and the fluid displacement field $\vec{\rho}$. This is perverse. Let ρ be the density and $\vec{u}(\vec{x})$ be the motion of the fluid element away from its initial position $\vec{x}$. Newton's law says

$$F = -\vec{\nabla} p\,\Delta x\,\Delta y\,\Delta z = ma = (\rho\,\Delta x\,\Delta y\,\Delta z)\frac{\partial^2 u}{\partial t^2}$$

$$\boxed{-\vec{\nabla} p = \rho \cdot \frac{\partial^2 u}{\partial t^2}}$$

← notice p and ρ
 pee rho

Now, the pressure $P - P_{atm} = p = -B\frac{\Delta V}{V}$.

The volume change $\frac{\Delta V}{V} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \vec{\nabla}\cdot\vec{u}$.

$$\boxed{p = -B\,\vec{\nabla}\cdot\vec{u}}$$

We can use the F=ma equation to eliminate $\vec{u}$ by taking the divergence of both sides:

$$-\vec{\nabla}\cdot\nabla p = \rho\frac{\partial^2(\vec{\nabla}\cdot\vec{u})}{\partial t^2} = \rho\frac{\partial^2(-p/B)}{\partial t^2}$$

$$\nabla^2 p = \frac{\rho}{B}\frac{\partial^2 p}{\partial t^2}$$

$$\boxed{\frac{\partial^2 p}{\partial t^2} = c^2\nabla^2 p \qquad \text{with } c = \sqrt{\frac{B}{\rho}}}$$

Here ∇^2 = "del-squared" = Laplacian = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

and we get the three-dimensional wave equation.

If things vary <u>only</u> along x, we get the one-dimensional sound wave equation we derived before.

Spherical Acoustic Waves

Exercise: Sound is emanating in a spherically symmetric way from a source. How should the amplitude of the pressure vary with distance $r = \sqrt{x^2+y^2+z^2}$ from the source? Hint: conservation of energy.

Surface area of sphere $= 4\pi r^2 \sim r^2$

Intensity = Power/Area

Power flowing out of sphere independent of r

$\rightarrow$ Intensity $\sim \frac{1}{r^2}$

Intensity $\sim P_{max}^2$ [remember $-\tau \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$, two y's]

$$\boxed{P_{max} \sim \frac{1}{r}}$$

Sound in three dimensions can form "spherical traveling waves," moving outward with velocity c.

Exercise: Guess the formula for the pressure $p(r,t)$, in terms of the shape of the wave f.

$$p(r,t) = \frac{1}{r} f(r-ct)$$

Does this solve the wave equation?

[The analogous formula in two dimensions does not! Ripples formed by dropping pebbles into ponds are <u>not</u> $h(r,t) = \frac{1}{\sqrt{r}} f(r-ct)$.]

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial^2}{\partial t^2} \frac{f(r-ct)}{r} = \frac{\partial}{\partial t}\left(-\frac{c f'(r-ct)}{r}\right) = c^2 \frac{f''(r-ct)}{r}.$$

So, we'd like $\nabla^2 p = \frac{f''(r-ct)}{r}$,

so $\frac{\partial^2 p}{\partial t^2} = \frac{c^2 f''}{r} = c^2 \nabla p^2$

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x} \frac{f(r-ct)}{r} = \left(\frac{f'}{r} - \frac{f}{r^2}\right)\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$= \frac{x}{r^3}\left(r f' - f\right)$$

$$\frac{\partial^2 p}{\partial x^2} = \left(\frac{1}{r^3} - \frac{3x}{r^4}\left(\frac{x}{r}\right)\right)(r f' - f) + \frac{x}{r^3}\left(\cancel{f'} + r f'' - \cancel{f'}\right)\left(\frac{x}{r}\right)$$

$$= \left(\frac{1}{r^3} - \frac{3x^2}{r^5}\right)(r f' - f) + \frac{x^2}{r^3} f''$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \overbrace{\left(\frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5}\right)}^{0}(r f' - f) + \frac{\overbrace{x^2+y^2+z^2}^{r^2}}{r^3} f''$$

$$= \frac{f''}{r}$$

So, $\frac{\partial^2 p}{\partial t^2} = c^2 \frac{f''}{r} = c^2 \nabla^2 p$ solves the 3D wave equation.