How to Invent New Laws
James P. Sethna, 2003
Latest Revision: September 24, 2003, 9:03

Ever since Newton, one of our primary jobs as scientists is to derive new laws. I'm referring here not to new Laws of Physics, the basic underpinning laws governing time, space, and matter*: only a few get to discover or invent these. Rather, we routinely invent physical laws with a small ell, governing the behavior of specific physical systems and depending upon their individual characteristics.

These small-ell laws *emerge* from the more fundamental, complicated laws, as we take certain limits and make use of conservation laws and small parameters. This is quite analogous to the way non-relativistic quantum mechanics emerges from quantum electrodynamics, which in turn emerges from various grand unified theories, one of which presumably emerges from string theory: at low energies, low frequencies, and long wavelengths, simpler theories emerge from more general but less useful ones. Christopher Henley, a colleague of mine here, suggests that one should think of each new material or physical system as a separate little universe with its own laws. We condensed-matter physicists get to play in thousands of different universes, where the particle physicists and cosmologists are stuck with studying just one.

We've had so much practise in inventing new theories that we've almost developed a kind of recipe. I'm describing the recipe used by condensed-matter physicists, but rather similar ones are used in various other fields of physics and engineering (using rather different vocabulary). Our methods were first used to guess the physical laws for superconductors, superfluids, and magnets by Landau, and were based on writing down the most general allowed form of the energy of a system allowed by symmetry. Landau theory has remained of great importance, but more recent work has started working not with the energy, but with the dynamical laws of motion themselves. These approaches allow one to study problems (like surface growth under atomic deposition, or crack growth) where energy minimization isn't the governing principle. We'll focus on these newer approaches.

Recipe applied to the Wave Equation for a Stretched String.

You may have noticed that we've derived the wave equation in two different contexts – transverse waves on strings and waves in air – and that the same wave equation also appears to describe light waves, longitudinal waves on springs, torsional wave machines, and a host of other systems. Doing a separate free-body diagram for each of these problems seems tedious. Is it possible to derive all (or perhaps lots) of these equations in one fell swoop? We'll argue that wave equations show up in all of these systems because they share certain symmetries, and that we can derive the wave equation – with a few assumptions and

* Or corresponding general laws of chemistry or economics.

choices – just by using these symmetries and focusing on long wavelength, low frequency behaviors. We break the problem up into several steps ...

(1) Pick an Order Parameter Field.

The order parameter field is a local variable that determines the properties of importance and interest for our system. For us, the vertical height $\eta(x,t)$ of the string is the natural order parameter.[†]

(2) Imagine the most general possible law.

For us, we want to know how $\eta(x,t)$ evolves in time. We expect a local law, so the evolution will involve η and various local slopes, velocities, and other derivatives. Whatever our equation is, we can subtract the right-hand side from the left hand side and get something of the form

$$\mathcal{F}(\eta, x, t, \partial\eta/\partial x, \partial\eta/\partial t, \partial^2\eta/\partial x^2, \partial^2\eta/\partial t^2, \partial^2\eta/\partial x\partial t, \ldots, \partial^7\eta/\partial x^3\partial t^4, \ldots) = 0. \qquad (NL1)$$

Here $\mathcal{F}$ is some general, ugly, nonlinear function.

(3) Restrict attention to long length and time scales.

We are large and slow creatures. Only the long-wavelength oscillations of the strings are low enough frequency for us to perceive: the vibrations with wavelengths comparable to the width of the string, for example, aren't really of much experimental interest.

This simplifies things a lot: the terms with high derivatives become small when you look on long length and time scales. If the string is wiggling with a characteristic length scale D, for example, the N^{th} derivative $\partial^N\eta/\partial x^N \sim 1/D^N$. When the wavelength or wiggle size D gets big, the derivative gets small and can be ignored (set to zero in $\mathcal{F}$).

Exercise. Consider a sine wave $\eta(x)$ of amplitude one and wavelength D. What is the function $\eta(x)$? (Get the 2π's correct!) What is its 400^{th} derivative $\partial^{400}\eta/\partial x^{400}$? Does it go as $1/D^{400}$?[*]

In our case, we'll keep terms with up to two partial derivatives. The kind of effects we lose with this approximation arise when wavelengths become comparable to the thickness of the string. So far our emergent equation of motion is

$$\mathcal{F}(\eta, x, t, \partial\eta/\partial x, \partial\eta/\partial t, \partial^2\eta/\partial x^2, \partial^2\eta/\partial t^2, \partial^2\eta/\partial x\partial t) = 0. \qquad (NL2)$$

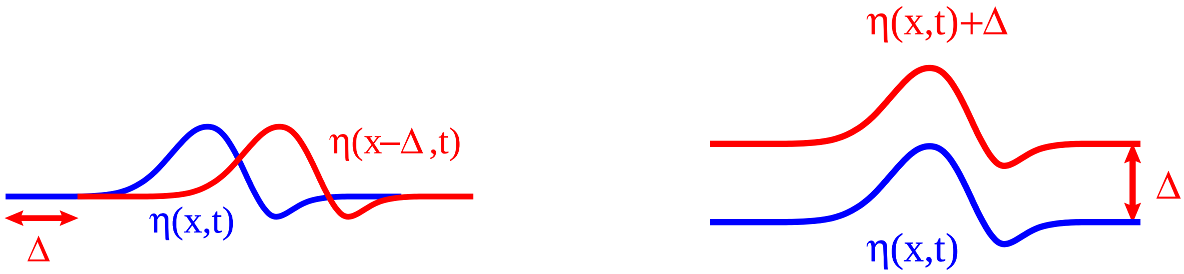
[†] It's called the "order parameter" because it often describes a kind of spontaneously developed order – like the magnetization field in a magnet. The name isn't really appropriate for waves on strings.

[*] Hint: the 4^{th} derivative of $\cos(\theta)$ is $\cos(\theta)$.

(4) Identify the symmetries of the system.

Assume an infinite, stretched, horizontal string, in vacuum (no losses) and without gravity. What symmetries does this system have? What do they imply about the solutions $\eta(x,t)$ of our equation of motion $\mathcal{F} = 0$?

Continuous symmetries.



The physical system of a wave on a string is homogeneous in space (left) and invariant under uniform vertical displacements (right).

Homogeneous (Translation along x). Our system is homogeneous in space: one part of the string looks the same (and obeys the same equations) as any other part. (There are no bulges, or knots, or changes in tension along the length of the string, for example.) Thus the string must obey the same equations if we translate it sideways. Specifically, if $\eta(x,t)$ is a solution, so also must be $\eta(x - \Delta)$ for any shift Δ to the right.‡

Time independent (Translation along t). Our system is time independent. (It wouldn't be, for example, if we applied a time-dependent force to it.) Thus if $\eta(x,t)$ is a solution, so also must be $\eta(x, t - \Delta)$ for any time shift Δ forward in time.

Sideways motion: Translation along y. Our system is the same if we shift the string vertically. Thus $\eta(x,t) + \Delta$ must also be a solution.

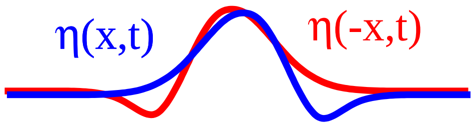
Discrete symmetries.

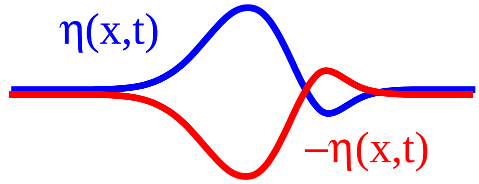
Parity: Reflection along x. The string looks the same if one flips it end-for end (reflecting through $x = 0$). If $\eta(x,t)$ is a solution, the $\eta(-x,t)$ must also be one.

Time-reversal invariance: Reflection along t. The (lossless) string looks the same if one reverses the sign of time, so $\eta(x,-t)$ must be a solution.

Inversion: Reflection along y. Inverting the string vertically is also a symmetry: $-\eta(x,t)$ is a solution.

‡ Obviously this messes up the boundary conditions, but we've assumed an infinite string with no boundaries.





The physical system of a wave on a string is invariant under reflections along x, perpendicular to the string (flips, left) and under reflections along the vertical y (inversions, right).

Our plan is now to generate the most general law of motion allowed by these symmetries, up to quadratic order in the gradients. If we do so, then there is an excellent chance that any new system we come up with that has our symmetries will, on long length and time scales, obey our law of motion.

(5) Apply the continuous symmetries.

The equation of motion $\mathcal{F} = 0$ should have the same continuous symmetries as the physical system has.

Sideways Motion. Let's plug in $\eta_0(x,t) = \eta(x,t) + \Delta$ into our equation of motion – η shifted vertically by Δ. We find that η_0 is a solution if it satisfies $\mathcal{F}(\eta + \Delta, x, t, \partial(\eta + \Delta)/\partial x, \ldots) = 0$. But Δ drops out of all of the partial derivatives, so it's a solution if $\mathcal{F}(\eta + \Delta, x, t, \partial\eta/\partial x, \ldots) = 0$. Clearly, we want our ugly, nonlinear function to be independent of the first argument (since adding any constant to the first argument doesn't affect the solutions)! So, because our system has a symmetry under sideways motion, we can assume $\mathcal{F}$ is not explicitly dependent on x.*

Homogeneous, Time Invariant. In a very similar way, we can show that $\mathcal{F}$ cannot explicitly depend upon x or t. Consider $\eta_0(x,t) = \eta(x - \Delta, t)$, shifting to the right by Δ. The argument is a bit trickier, because $\mathcal{F}$ applied to η_0 is the same as $\mathcal{F}$ applied to η not at the same point, but at *corresponding* points. That is, $x + \Delta$ for η_0 corresponds to x for η, so

$$\mathcal{F}(\eta_0(x+\Delta,t), x+\Delta, t, \partial\eta_0(x+\Delta,t)/\partial x, \ldots) = \mathcal{F}(\eta(x,t), x, t, \partial\eta(x,t)/\partial x, \ldots).$$

But, of course, $\eta_0(x+\delta,t) = \eta(x,t)$, so we find again $\mathcal{F}(\eta, x+\Delta, t, \ldots) = \mathcal{F}(\eta, x, t, \ldots)$ and so $\mathcal{F}$ can't depend explicitly on x. Similarly, $\mathcal{F}$ can't depend explicity on time, because our system is time independent.

Our equation of motion has lost its first three arguments, and now simplifies to

$$\mathcal{F}(\partial\eta/\partial x, \partial\eta/\partial t, \partial^2\eta/\partial x^2, \partial^2\eta/\partial t^2, \partial^2\eta/\partial x\partial t) = 0. \qquad (NL3)$$

* It can of course still depend on derivatives of η, which are unchanged by a constant shift.

(6) (Often) Assume that the order parameter is small.

We often may assume the order parameter is small, and keep only terms with low powers of it. (This corresponds to the small angle approximation in our derivation of waves on strings, and the assumption that the compressions $\frac{\Delta V}{V}$ were small in the derivation for sound waves.) In our case, we'll keep terms only to linear order in η. This gives us a linear equation with only seven unknown parameters $A, B, C, D, E, F \ldots$

$$A + B\partial\eta/\partial x + C\partial\eta/\partial t + D\partial^2\eta/\partial x^2 + E\partial^2\eta/\partial t^2 + F\partial^2\eta/\partial x\partial t = 0. \qquad (NL4)$$

(7) Apply the discrete symmetries.

We argued that $\mathcal{F}$ must remain unchanged under continuous symmetries. For discrete symmetries like reflection or time-reversal, though, we discover that we have a choice. $\mathcal{F}$ may be unchanged under the symmetry, or it may change sign. (Since -0 = 0, the equation $\mathcal{F} = 0$ is still valid if $\mathcal{F}$ changes sign.) Let's see how this plays out for each of the three discrete symmetries.

Inversion: $y \rightarrow -y$**.** Under this symmetry, $\eta_0(x,t) = -\eta(x,t)$. All terms linear in η in equation (NL4) change sign (that is, all terms other than the constant term A). If we decide to choose $\mathcal{F}$ to change sign under reflection, we must insist that $A = 0$; otherwise we must choose all the other constants equal to zero. Let's go with the first choice.[†] We deduce that $A = 0$.

Flipping: $x \rightarrow -x$**.** Under this symmetry, $\eta_0(x,t) = \eta(-x,t)$. It's easy to see graphically that the slope (first partial derivatives η_0 with respect to x) changes sign, but the curvature (second partial derivative with respect to x) does not. Using the chain rule, we see $\frac{\partial\eta_0}{\partial x}(x,t) = -\frac{\partial\eta}{\partial x}(-x,t)$, $\frac{\partial\eta_0}{\partial t}(x,t) = \frac{\partial\eta}{\partial t}(-x,t)$, $\frac{\partial^2\eta_0}{\partial x^2}(x,t) = \frac{\partial^2\eta}{\partial x^2}(-x,t)$, and so on. Generally, terms odd in $\partial/\partial x$ change sign, and terms even stay unchanged. If we choose $\mathcal{F} \rightarrow \mathcal{F}$ under flipping, we must set to zero the two terms odd in $\partial/\partial x$, so $B = F = 0$

Time reversal: $t \rightarrow -t$**.** Under this symmetry, $\eta_0(x,t) = \eta(x,-t)$. This implies that the term odd in $\partial/\partial t$ vanishes, so $C = 0$ as above. We arrive at the following general equation

$$D\partial^2\eta/\partial x^2 + E\partial^2\eta/\partial t^2 = 0. \qquad (NL5)$$

so

$$\partial^2\eta/\partial t^2 = -D/E\partial^2\eta/\partial x^2. \qquad (NL6)$$

If we choose $D/E < 0$, we get the wave equation!

[†] If there is a system with our symmetries where $\mathcal{F}$ doesn't change sign under $y \rightarrow -y$, it's described by a nonlinear differential equation quadratic in η.