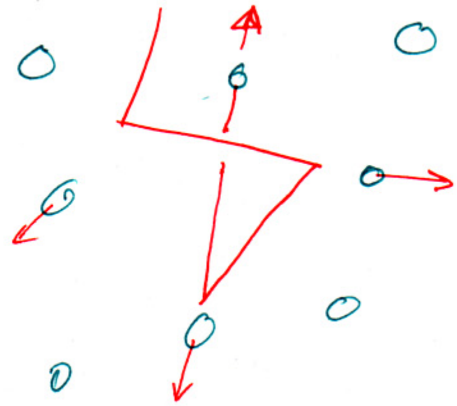
Random Walks

- Molecules diffusing through the air undergo a random walk: straight lines between collisions

- Density of Molecules
  =(Probability)(Number)



- Derive
  $$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho$$
  from random walks?

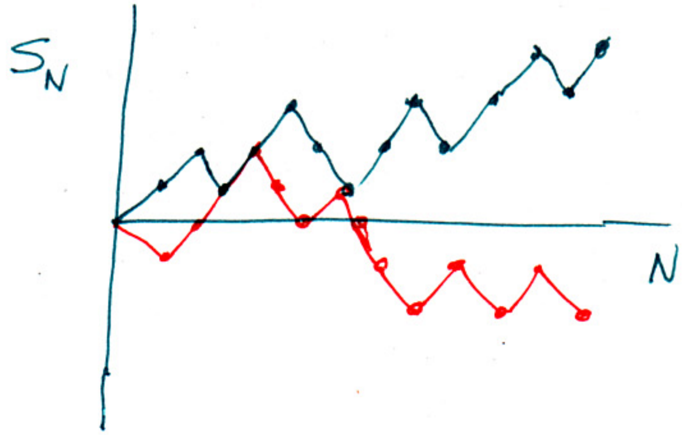
Free Body Diagram

Coin Flips

N coin flips

$$x_i = \begin{matrix} +1 & \text{heads} \\ -1 & \text{tails} \end{matrix}$$

$$S_N = \sum_{i=1}^{N} x_i$$



How far does it get after N steps?

$$\langle S_N \rangle = 0 = \sum_{i=1}^{N} \langle x_i \rangle$$

Positive & Negative Cancel

$\sqrt{\langle S_N^2 \rangle}$ Better measure of typical distance

<u>Induction</u>

$$\langle S_N^2 \rangle = \langle (S_{N-1} + x_N)^2 \rangle = \langle S_{N-1}^2 \rangle + 2 \langle S_{N-1} x_N \rangle + \langle x_N^2 \rangle$$

$$= \langle S_{N-1}^2 \rangle + 2 \Big(\langle S_{N-1} \rangle (+1) + \langle S_{N-1} \rangle (-1) \Big) + 1$$

$$= \langle S_{N-1}^2 \rangle + 1 = \ldots = N$$

Average root-mean-square distance moved $= \sqrt{N}$

"RMS"

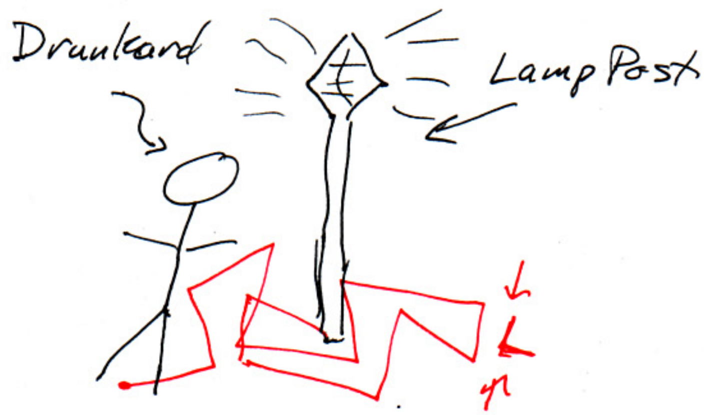
Just like diffusion equation!

Probability distribution

$$\rho(s, N) = \frac{1}{2} \rho(s+1, N-1) + \frac{1}{2} \rho(s-1, N-1)$$

Will return later! Link to $\partial \rho / \partial t = D \nabla^2 \rho$!

Drunkard's Walk



- Starts at Lamppost $\vec{S}_0 = \vec{0}$
- Each step $\vec{X}_N$ random direction, length L

$$\vec{S}_N = \vec{S}_{N-1} + \vec{X}_N$$

$$\langle \vec{S}_N \rangle = 0$$

$$\langle \vec{S}_N^{\,2} \rangle = \langle (\vec{S}_{N-1} + \vec{X}_N)^2 \rangle$$

$$= \langle (S_{N-1})^2 \rangle + \overset{0}{\cancel{2 \langle \vec{S}_{N-1} \cdot \vec{X}_N \rangle}} + \overset{L^2}{\langle X_N^2 \rangle}$$

$\vec{X}_N$ random direction, cancels $-\vec{X}_N$ on averaging

$$= N L^2$$

RMS Distance $\sim \sqrt{N}\, L$

General Case

Let S_N be the sum of N random variables x_i.

Let each x_i have the same probability distribution $\xi(x)$, with zero mean and RMS $= a$:

$$\int \xi(x) = 1, \quad \int x\,\xi(x) = 0,$$

$$\int x^2 \xi(x) = a^2$$

Then $\langle s_N \rangle = 0$ and

$$\langle s_N^2 \rangle = \langle (s_{N-1} + x_N)^2 \rangle$$

$$= \langle s_{N-1}^2 \rangle + \cancel{2\langle s_{N-1} x_N \rangle} + \langle x_N^2 \rangle$$

Independent

$$= \langle s_{N-1}^2 \rangle + a^2 = Na^2$$

and the evolution law for ρ is

$$\rho(s,N) = \int \underbrace{\rho(s', N-1)}_{\text{How likely at } s' \text{ before}} \underbrace{\xi(s-s')}_{\text{How likely last step went from } s' \text{ to } s} ds'$$

The Continuum Limit & the Diffusion Equation

- Suppose the time between steps is Δt.

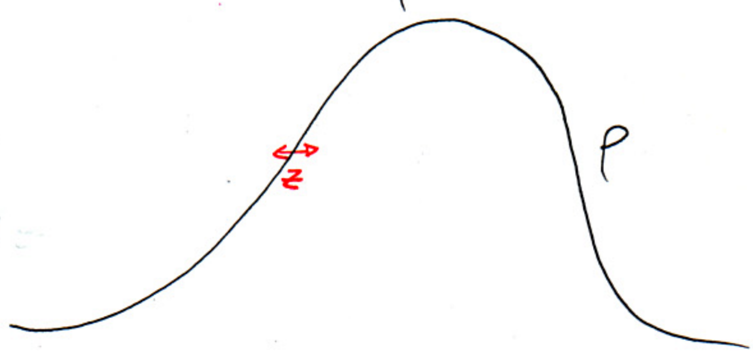
$$s(t+\Delta t) = s(t) + x(t)$$

$$\rho(s, t+\Delta t) = \int \rho(s', t)\, \xi(s-s')\, ds' \qquad z = s - s' \quad dz = -ds'$$

$$= \int \rho(s-z, t)\, \xi(z)\, dz$$

- Suppose that the step size is very small compared to the sizes of variation in ρ

→ Taylor expand in z



$$\rho(s, t+\Delta t) = \int \left[\rho(s,t) - z\frac{\partial\rho}{\partial s} + \frac{z^2}{2}\frac{\partial^2\rho}{\partial s^2}\right] \xi(z)\, dz$$

$$= \rho(s,t) \overbrace{\int \xi(z)\, dz}^{1} - \frac{\partial\rho}{\partial s} \overset{0}{\cancel{\int z\, \xi(z)\, dz}}$$

$$+ \frac{\partial^2\rho}{\partial s^2}\, \frac{1}{2} \underbrace{\int z^2 \xi(z)\, dz}_{a^2}$$

So, $\frac{\partial \rho}{\partial t} \Delta t = \frac{a^2}{2} \frac{\partial^2 \rho}{\partial x^2}$

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} = \frac{a^2}{2\Delta t} \frac{\partial^2 \rho}{\partial x^2}}$$

"Microscopic" Derivation of Diffusion Equation

Consequences:

(1) $D = a^2/2\Delta t$. Can estimate diffusion constant from collision lengths and times

(2) Random walks, sums of random variables, become Gaussian in continuum limit

$$\boxed{\text{Central Limit Theorem}}$$

$$\rho_N(x) \rightarrow \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$$

Gaussian also called "Normal" distribution

as $N \rightarrow \infty$, with $\sigma^2 = N a^2$

["Almost perfect" for $N \gtrsim 5$]