Solving the Diffusion Equation

$$\frac{\partial \rho}{\partial t} = D \nabla^2 \rho$$

- First-order in time
  - → specify initial value $\rho(x,0)$
  - [no "initial velocities"]

"Standing Wave" Solutions?

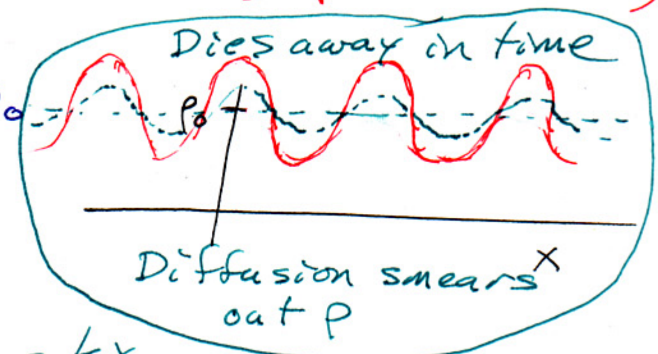
Homogeneous in Space ⇒ Fourier Methods

$f_k(x) = e^{ikx}$ $\quad f_k(x+\Delta) = e^{ik\Delta} f_k(x)$

→ eigenfunctions of translation

→ all linear, homogeneous systems have plane wave solutions (lecture Fourier)

$$\rho(x,t) = A(t)\cos(kx) + \rho_0$$



$$\frac{\partial \rho}{\partial t} = \frac{dA}{dt}\cos kx$$

$$= D\nabla^2 \rho = -Dk^2 A(t)\cos kx$$

$$\frac{dA}{dt} = -Dk^2 A \qquad A(t) = A_0 e^{-Dk^2 t} \cancel{+\rho_0}$$

Similarly, $\rho(x,t) = \rho_k e^{ikx} e^{-Dk^2 t}$ is a solution,
→ so is

$$\boxed{\rho(x,t) = \underset{?}{\underline{\frac{1}{2\pi}}} \int \tilde{\rho}_k(0) e^{ikx} e^{-Dk^2 t}\, dk}$$ General Solution

with $\boxed{\tilde{\rho}_k(0) = \int \rho(x,0)\, e^{-ikx}\, dx}$

Whence the 2π? In a box of size L,

$$\rho(x,t) = \frac{1}{L} \sum_n \tilde{\rho}_{k_n} e^{ik_n x}$$

$$k_n = \frac{2\pi n}{L} = n\,\Delta k, \quad \Delta k = 2\pi/L, \quad \frac{1}{L} = \frac{1}{2\pi}\Delta k$$

$$\rho(x,t) = \frac{1}{2\pi} \sum_n \tilde{\rho}_{k_n} e^{ik_n x} \Delta k \underset{L\to\infty}{\longrightarrow} \frac{1}{2\pi} \int \tilde{\rho}_k e^{ikx}\, dk$$

- Just like the wave equation: completely general, but mostly useless? Good for numerics, though!
- Is there an analogy to travelling waves, where we can draw pictures & generate intuition?

Suppose I have one unit of perfume, released at $x=0$ at time $t=0$. What is $\rho(x,0)$?

$$\rho(x,0) = \begin{cases} \infty & \text{at } x=0 \\ 0 & x \neq 0 \end{cases} \qquad \int \rho(x,0)\,dx = 1$$

Obviously, this is a weird (but useful) function. We call it $\delta(x)$, the Dirac delta function:

$$\delta(x) = 0 \qquad x \neq 0$$

$$\int f(x)\,\delta(x)\,dx = f(0) \rightarrow \int f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

What is $\rho(x,t)$, if $\rho(x,0) = \delta(x)$? Use Fourier transforms!

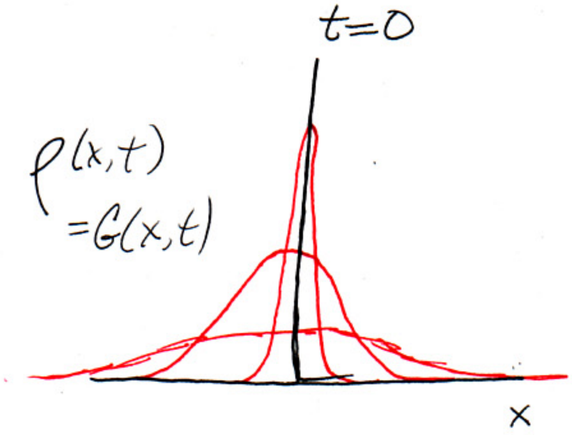
$$\tilde{\rho}_k(0) = \int \delta(x)\, e^{-ikx}\, dx = 1 \quad \left(\begin{matrix}\text{independent}\\ \text{of } k\end{matrix}\right)$$

$$\rho(x,t) = \frac{1}{2\pi}\int \tilde{\rho}_k(0)\, e^{ikx} e^{-Dk^2t}\, dk$$

$$= \underbrace{\frac{1}{2\pi}\int e^{ikx} e^{-Dk^2t}\, dk}_{\text{Fourier Transform of a Gaussian is a Gaussian}}$$

$e^{-\sigma^2k^2/2} \rightarrow$ width $\sigma^2 = 2Dt$

$$\boxed{\rho(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-x^2/4Dt} = G(x,t)}$$



Perfume spreads out to distances

$x \sim \sqrt{t}$

RMS Width $\sigma = \sqrt{2Dt}$

Height $= \dfrac{1}{\sqrt{2\pi}\,\sigma}$

Can we build a general solution from this one?

Initial $\rho(x,0)$ "smears out" by range $x \sim \sqrt{2Dt}$?

We can write

$$\rho(x,0) = \int \rho(y,0)\, \delta(y-x)\, dy$$

$z = y - x$

$\int \rho(z+x)\, \delta(z)\, dz$ evaluates at $z=0$

$\rho(x)$

so

$$\rho(x,t) = \int \rho(y,0)\, G(y-x,t)\, dy$$

$$= \int \rho(y,0) \frac{e^{-\frac{(y-x)^2}{4Dt}}}{\sqrt{4\pi Dt}}\, dy$$

General solution

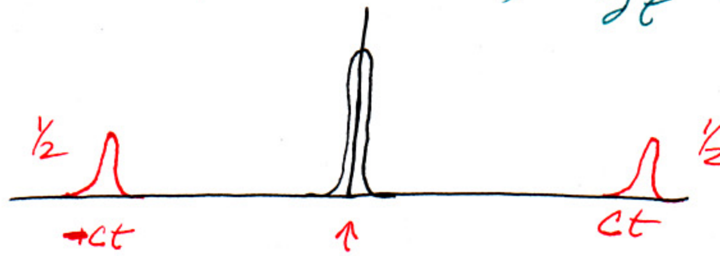
Current value $=$ (Original neighborhood) (Smeared sideways by G)

G is called the Green's function for the problem.

Why is this analogous to travelling waves?

G for the wave equation:

$$\eta(x,0) = \delta(x), \quad \frac{\partial \eta}{\partial t}(x,0) = 0$$



Pulse is superposition of half-size pulses to left & right

Easier to imagine $\eta(x,0)$ spread out a tiny bit

$$G(x,t) = \frac{1}{2}\delta(x+ct) + \frac{1}{2}\delta(x-ct)$$

General solution

$$\eta(x,t) = \int \eta(y,0)\, G(y-x,t)\, dy$$

$$= \int \eta(y,0)\big(\delta(x+ct)+\delta(x-ct)\big)/2\, dy$$

$$= \underbrace{\frac{1}{2}\eta(x-ct,0) + \frac{1}{2}\eta(x+ct,0)}$$

Travelling waves to left & right
Same shape as original, half height!

Examples of Diffusion Equations

Diffusion of Matter
(Coffee, Perfume, Neutrons, ...)
D = Diffusion Constant

Diffusion of Heat

$$\frac{\partial T}{\partial t} = \frac{K}{c} \nabla^2 T$$

Problem Set 10

K = Thermal Conductivity
c = Specific Heat

Diffusion of Photons through the Sun

Diffusion of Electrons in Dirty Metals
⇒ OHM's Law, ...

Diffusion of Cell Orientations in
Rotation-Matrix Space ...