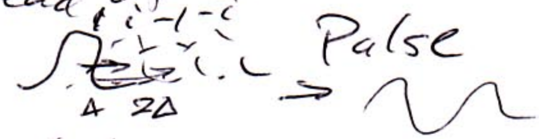
	$\mathbb{R}^3$	$\mathbb{S}$
	Space of Vectors (Positions)	Space of Functions $y(x+L) = y(x)$
Domain	$\vec{r} = (r_1, r_2, r_3)$ one real per $\{1, 2, 3\}$	$y(x)$, $\eta(x)$, $\xi(x)$ one real per $x \in (0, L)$
Dot Product	$\vec{r} \cdot \vec{s} = r_1 s_1 + r_2 s_2 + r_3 s_3$	$\eta \cdot \xi = \frac{1}{L} \int_0^L \eta(x)\, \xi^*(x)\, dx$
Distance	$\lvert \vec{r} - \vec{s} \rvert = \sqrt{(\vec{r} - \vec{s})^2}$	$\lVert \eta - \xi \rVert_2 = \sqrt{\frac{1}{L} \int_0^L \lvert \eta(x) - \xi(x) \rvert^2 dx}$
Basis (Unit Vectors)	$\hat{x}_1, \hat{x}_2, \hat{x}_3$ $(\hat{x}, \hat{y}, \hat{z})$	$\hat{f}_m = e^{i k_m x}$ $\quad k_m = \frac{2m\pi}{L}$
Norm 1 Orthogonal	$\hat{x}_i^{\,2} = 1$ $\hat{x}_i \cdot \hat{x}_j = 0$	$\frac{1}{L} \int_0^L e^{i k_m x} e^{-i k_n x} dx = \begin{cases} 1 & m = n \\ 0 & \text{otherwise} \end{cases}$
Coefficients	$r_n = \vec{r} \cdot \hat{x}_n$	$\tilde{\eta}_m = \eta \cdot \hat{f}_m = \frac{1}{L} \int_0^L \eta(x)\, e^{-i k_m x} dx$
Completeness (Got all the directions)	$\vec{r} = \sum_1^3 r_n \hat{x}_n$	$\eta(x) = \sum \tilde{\eta}_m \hat{f}_m(x) = \sum_m \tilde{\eta}_m e^{i k_m x}$

Next Year!
X instead of t

Question 2:



Why are the solutions cosines and sines?

Because $e^{i\omega t} = \cos\omega t + i\sin\omega t$

Why is $e^{i\omega t}$ special?

It's the eigenfunction of translations.

Translational Symmetries:

if $y(x,t)$ is a solution, so is

Time Independent $T_\Delta(y) = y(x, t-\Delta)$

Homogeneous $R_\Delta(y) = y(x-\Delta, t)$

- T_Δ and R_Δ shift functions to the right in time and space, by amount Δ
- Equations of motion respect these symmetries
- T_Δ and R_Δ are <u>linear</u> mappings from $\mathcal{S}$ to $\mathcal{S}$

Analogy to $\mathbb{R}^3$: linear mappings are 3x3 matrices $\vec{r} \to M\cdot\vec{r} = \sum_j M_{ij} r_j$

Eigenvectors of M are often useful $M\cdot\vec{e}_n = \lambda_n \vec{e}_n$

What are the eigenfunctions of T_Δ?

$$T_\Delta(f_\omega) = f_\omega(t-\Delta) = \lambda_\omega f(t)$$

$$\boxed{f_\omega(t) = e^{i\omega t}}$$

$$f_\omega(t-\Delta) = e^{i\omega(t-\Delta)} = e^{-i\omega\Delta} f_\omega(t)$$

$$\boxed{\lambda_\omega = e^{-i\omega\Delta}}$$

Can we use these to prove that there are solutions of the form $\eta_\omega(x,t) = e^{i\omega t} g(x)$?

Real, imaginary parts give $\cos(\omega t)$, $\sin(\omega t)$ solutions.

Start with <u>any</u> solution $\eta(x,t)$.

$\eta(x, t-\Delta) = T_\Delta(\eta)$ is also a solution for <u>all</u> Δ.

Superimpose these solutions, ~~dividing~~ dividing by λ_ω

the complex

$$\eta_\omega(x,t) = \int_{-\infty}^{\infty} \eta(x, t-\Delta)/e^{-i\omega\Delta}\, d\Delta$$

$$= \int_{-\infty}^{\infty} e^{i\omega\Delta}\, \eta(x, t-\Delta)\, d\Delta \qquad \xi = t-\Delta \quad \Delta = t-\xi \quad d\xi = -d\Delta$$

$$= \int_{-\infty}^{\infty} e^{i\omega(t-\xi)}\, \eta(x,\xi)\, [-d\xi] \qquad \int_{-\infty}^{\infty} \to \int_{\infty}^{-\infty}$$

Complex version of Fourier coefficient

$$= e^{i\omega t} \underbrace{\left[+ \int_{-\infty}^{\infty} e^{-i\omega\xi}\, \eta(x,\xi)\, d\xi \right]}_{\cancel{g(x)}\ \tilde{\eta}(x,\omega)}$$

Similarly, for an infinite string, can prove that there are solutions e^{ikx}!