Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
In Class Exercises
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1. **Self-similar sounds.**
   Explore sound files on course Web site http://pages.physics.cornell.edu/~sethna/teaching/653/HW.html

2. **Ising self-similarity.** ⓘ

   Start up the Ising model (computer exercises portion of the book web site [1]). Run a large system at zero external field and $T = T_c = 2/\log(1+\sqrt{2}) \approx 2.26919$. Set the refresh rate low enough that graphics is not the bottle-neck, and run for at least a few hundred sweeps to equilibrate. You should see a fairly self-similar structure, with fractal-looking up-spin clusters inside larger down-spin structures inside …

   Can you find a nested chain of three clusters? Four?

3. **Random Walks.** (Scaling) ⓘ

   Self-similar behavior also emerges without proximity to any obvious transition. One might say that some *phases* naturally have self-similarity and power laws. Mathematicians have a technical term *generic* which roughly translates to 'without tuning a parameter to a special value', and so this is termed *generic scale invariance*.

   The simplest example of generic scale invariance is that of a random walk. Figure 2.2 shows that a random walk appears statistically self-similar.

   Let $X(T) = \sum_{t=1}^{T} \xi_t$ be a random walk of length T, where ξ_t are independent random variables chosen from a distribution of mean zero and finite standard deviation. Derive the exponent ν governing the growth of the root-mean-square end-to-end distance $d(T) = \sqrt{\langle (X(T) - X(0))^2 \rangle}$ with T. Explain the connection between this and the formula from freshman lab courses for the way the standard deviation of the mean scales with the number of measurements.

4. **Superconductivity and the renormalization group.** (Condensed matter) ⓘ

   Ordinary superconductivity happens at a rather low temperature; in contrast to phonon energies (hundreds of degrees Kelvin times k_B) or electronic energies (tens of thousands of degrees Kelvin), phonon-mediated superconductivity in most materials happens below a few Kelvin. This is largely explained by the BCS theory of superconductivity, which predicts that the transition temperature for weakly-coupled superconductors is

   $$T_c = 1.764\,\hbar\omega_D \exp\left(-1/V g(\varepsilon_F)\right), \tag{1}$$

where ω_D is a characteristic phonon frequency, V is an attraction between electron pairs mediated by the phonons, and $g(\varepsilon_F)$ is the density of states (DOS) of the electron gas (eqn 7.74) at the Fermi energy. If V is small, $\exp\left(-1/V g(\varepsilon_F)\right)$ can be exponentially small, explaining why materials often have to be so cold to go superconducting.

Superconductivity was discovered decades before it was explained. Many looked for explanations which would involve interactions with phonons, but there was a serious obstacle. People had studied the interactions of phonons with electrons, and had shown that the system stays metallic (no superconductivity) *to all orders in perturbation theory.*

(a) *Taylor expand T_c (eqn 1) about $V = 0^+$ (about infinitesimal positive V). Guess the value of all the terms in the Taylor series. Can we expect to explain superconductivity at positive temperatures by perturbing in powers of V?*

There are two messages here.

- Proving something to all orders in perturbation theory does not make it true.
- Since phases are regions in which perturbation theory converges (see Section 8.3), the theorem is not a surprise. It is a condition for a metallic phase with a Fermi surface to exist at all.

In recent times, people have developed a renormalization-group description of the Fermi liquid state and its instabilities[1] (see note 23 on p. 144). Discussing Fermi liquid theory, the BCS theory of superconductivity, or this renormalization-group description would take us far into rather technical subjects. However, we can illustrate all three by analyzing a rather unusual renormalization-group flow.

Roughly speaking, the renormalization-group treatment of Fermi liquids says that the Fermi surface is a fixed-point of a coarse-graining in *energy*. That is, they start with a system space consisting of a partially-filled band of electrons with an energy width W, including all kinds of possible electron–electron repulsions and attractions. They coarse-grain by perturbatively eliminating (integrating out) the electronic states near the edges of the band,

$$W' = (1 - \delta)W, \tag{2}$$

incorporating their interactions and effects into altered interaction strengths among the remaining electrons. These altered interactions give the renormalization-group flow in the system space. The equation for W gives the change under one iteration ($n = 1$); we can pretend n is a continuous variable and take $\delta n \to 0$, so $(W' - W)/\delta \to \mathrm{d}W/\mathrm{d}n$, and hence

$$\mathrm{d}W/\mathrm{d}n = -W. \tag{3}$$

When they do this calculation, they find the following.

[1]There are also other instabilities of Fermi liquids. Charge-density waves, for example, also have the characteristic $\exp(-1/aV)$ dependence on the coupling V.

- The non-interacting Fermi gas we studied in Section 7.7 is a *fixed point of the renormalization group*. All interactions are zero at this fixed-point. Let V represent one of these interactions.[2]
- The fixed-point is unstable to an attractive interaction $V > 0$, but is stable to a repulsive interaction $V < 0$.

- Attractive forces between electrons grow under coarse-graining and lead to new phases, but repulsive forces shrink under coarse-graining, leading back to the metallic free Fermi gas.

  This is quite different from our renormalization-group treatment of phase transitions, where *relevant* directions like the temperature and field were unstable under coarse-graining, whether shifted up or down from the fixed-point, and other directions were *irrelevant* and stable (Fig. 12.8). For example, the temperature of our Fermi gas is a relevant variable, which rescales under coarse-graining like

$$\begin{aligned} T' &= (1 + a\delta)T, \\ \mathrm{d}T/\mathrm{d}n &= aT. \end{aligned} \tag{4}$$

  Here $a > 0$, so the effective temperature becomes larger as the system is coarse-grained. How can they get a variable V which grows for $V > 0$ and shrinks for $V < 0$?

- When they do the coarse-graining, they find that the interaction V is *marginal*: to linear order it neither increases nor decreases. The next allowed term in the Taylor series near the fixed-point gives us the coarse-grained equation for the interaction:

$$\begin{aligned} V' &= (1 + b\delta V)V, \\ \mathrm{d}V/\mathrm{d}n &= bV^2. \end{aligned} \tag{5}$$

- They find $b > 0$.

[2] V will be the pairing between opposite-spin electrons near the Fermi surface for superconductors.

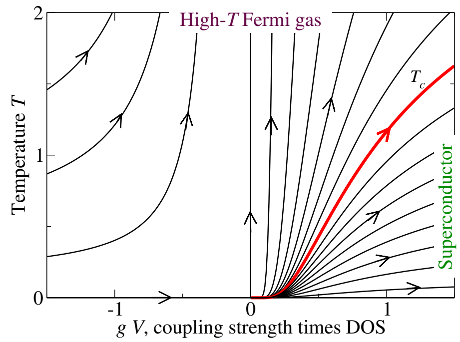

Fig. 1 Fermi liquid theory renormalization-group flows. The renormalization flows defined by eqns 4 and 5. The temperature T is relevant at the free Fermi gas fixed-point; the coupling V is marginal. The distinguished curve represents a phase transition boundary $T_c(V)$. Below T_c, for example, the system is superconducting; above T_c it is a (finite-temperature) metal.

(b) *True or false? (See Fig. 1.)*

(T) (F) For $V > 0$ (attractive interactions), the interactions get stronger with coarse-graining.

(T) (F) For $V < 0$ (repulsive interactions), coarse-graining leads us back to the free Fermi gas, explaining why the Fermi gas describes metals (Section 7.7).

(T) (F) Temperature is an irrelevant variable, but dangerous.

(T) (F) The scaling variable

$$x = TV^{1/\beta\delta} \tag{6}$$

is unchanged by the coarse-graining (second equations in 4 and 5), where β and δ are universal critical exponents;[3] *hence x labels the progress along the curves in Fig. 1 (increasing in the direction of the arrows).*

(T) (F) The scaling variable

$$y = T \exp\left(a/(bV)\right) \tag{7}$$

is unchanged by the coarse-graining, so each curve in Fig. 1 has a fixed value for y.

Now, without knowing anything about superconductivity, let us presume that our system goes superconducting at some temperature $T_c(V)$ when the interactions are attractive. When we coarse-grain a system that is at the superconducting transition temperature, we must get another system that is at its superconducting transition temperature.

(c) *What value for a/b must they calculate in order to get the BCS transition temperature (eqn 1) from this renormalization group? What is the value of the scaling variable (whichever you found in part (b)) along $T_c(V)$?*

[3] Note that here δ is not the infinitesimal change in parameter.

Thus the form of the BCS transition temperature at small V, eqn 1, can be explained by studying the Fermi gas *without reference to the superconducting phase*!

References

[1] Sethna, J. P. and Myers, C. R. (2004). *Entropy, Order Parameters, and Complexity* computer exercises: Hints and software. http://www.physics.cornell.edu/sethna/StatMech/ComputerExercises.html.