Physics 7653: Statistical Physics
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In Class Exercises
Last correction at September 5, 2018, 9:54 am

12.26 **The onset of chaos: Lowest order renormalization-group for period doubling.** (Dynamical systems)③

In this exercise, we set up a low-order approximate renormalization group to study the period-doubling route to chaos of Fig. 12.17. Our goal is to estimate the scaling factors $\alpha \sim -2.5029$ and $\delta \sim 4.6692$ governing the self-similarity in space x and control parameter μ near the onset of chaos μ_∞, as discussed in Exercise 12.22. We shall extend the renormalization group we develop here to higher order in Exercise 12.27.

The period-doubling route to chaos is understood by a renormalization group that (as usual) has a coarse-graining step and a rescaling. The coarse-graining step 'decimates' the time series $\{x, g(x), g(g(x)), \dots\}$ by dropping every other point, replacing the function $g(x)$ by $g(g(x))$. The rescaling expands x about its maximum by a factor α. If we choose the maximum value of $g(x)$ to be at zero, the renormalization group sends g to the function $T[g]$, where

$$T[g](x) = \alpha g\left(g(x/\alpha)\right). \tag{1}$$

(We explore the scaling and universality implied by this renormalization group in Exercise 12.9; our eqn 1 is the same as eqn 12.40) in that exercise, except there our functions were symmetric about $x = 1/2$.)

Our functions $g(x)$ are 'one-humped maps', with a parabolic maximum at zero. To lowest order, let us approximate $g(x)$ by a parabola centered at the origin

$$g(x) \approx G_0 + G_1 x^2. \tag{2}$$

where $G_1 < 0$.

We shall approximate the fixed point g^* of our renormalization group by demanding that $T[g^*](x) = g^*(x)$ at two points, $x = 0$ and $x = 1$. We have three constants, α, G_0^*, and G_1^*, so for convenience we set G_0^* to one.

(a) *Use the fixed-point condition at $x = 0$ to show that $\alpha = 1/g^*(1) = 1/(1 + G_1^*)$.*

(b) *Use the fixed-point condition at $x = 1$ to give an equation for G_1^*, substituting in your equation for α above.* (Hint: The equation simplifies to a sixth-order polynomial.)

We expect $\alpha \approx -2.5$, so since our approximate $\alpha = 1/(1 + G_1^*)$, we expect $G_1^* \approx 1/\alpha - 1 \approx -1.4$.

(c) *Plot the quantity from part (b) that must be zero: does it have a root near $G_1 = -1.4$? Numerically solve your equation from part (b) for the root closest to -1.4. What is α*

in our approximation? (Your approximation for α should be within a few percent of the correct value $\alpha = -2.5029\ldots$; your value for G_1^* should be within a few percent of -1.4 and not too far from the true value of the quadratic term at the fixed point, $-1.5276\ldots$.)

In statistical mechanics, we find the universal critical exponents by linearizing the renormalization-group flows about the fixed point and finding directions that grow. Here the exponent $\delta = 4.669\ldots$ describes the fastest growing direction in function space: $T[g^* + \epsilon\psi](x) - g^*(x) = \delta\psi(x)$. That is, we add a perturbation $g(x) = g^*(x) + \epsilon\psi(x)$ and study to linear order in ϵ how the perturbation grows under T. The lowest-order perturbation to our parabola adds an overall constant $G_0 \to 1 + \epsilon$. Our function $g^*(x)$ is fixed at both $x = 0$ and $x = 1$. Let us check the growth of our perturbation at $x = 0$, which should grow by a factor of approximately δ when our renormalization-group transformation is applied.

(d) *Using $g(x) = (1+\epsilon) + G_1 x^2$, write the formula for the term in $T[g](0)$ linear in ϵ as a function of G_1 and α. Insert your fixed-point values for G_1 and α from part (c). What is your estimate for δ?* (Your approximation for δ should be within a few percent of the correct value $\delta = 4.669\ldots$.)

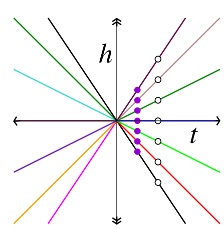
12.7 **Renormalization-group trajectories.** ⓟ

This exercise provides an early introduction to how we will derive power laws and universal scaling functions in Section 12.2 from universality and coarse-graining.

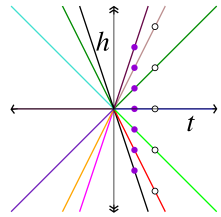
An Ising model near its critical temperature T_c is described by two variables: the distance to the critical temperature $t = (T - T_c)/T_c$, and the external field $h = H/J$. Under coarse-graining, changing lengths to $x' = (1 - \epsilon)\,x$, the system is observed to be similar to itself at a shifted temperature $t' = (1 + a\epsilon)\,t$ and a shifted external field $h' = (1 + b\epsilon)\,h$, with ϵ infinitesimal and $a > b > 0$ (so there are two relevant eigendirections, with the temperature more strongly relevant than the external field). Assume $a > b > 0$.

The curves shown below connect points that are similar up to some rescaling factor.

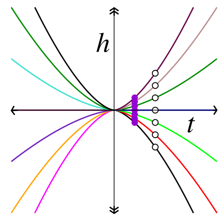
(a) *Which diagram below has curves consistent with this flow, for $a > b > 0$? Is the flow under coarse graining inward or outward from the origin?* (No math should be required. Hint: After coarse-graining, how does h/t change?)



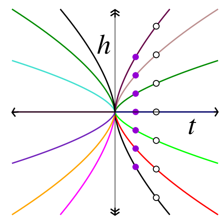
(A)



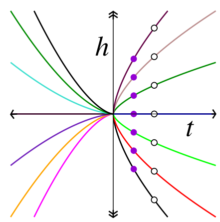
(B)



(C)



(D)



(E)

The solid dots are at temperature t_0; the open circles are at temperature $t = 2t_0$.

(b) *In terms of* ϵ *and* a*, by what factor must* x *be rescaled by to relate the systems at* t_0 *and* $2t_0$*?* (Algebraic tricks: Use $(1+\delta) \approx \exp(\delta)$ everywhere. If you rescale multiple times until $\exp(na\epsilon) = 2$, you can solve for $(1-\epsilon)^n \approx \exp(-n\epsilon)$ without solving for n.) *If one of the solid dots in the appropriate figure from part (a) is at* (t_0, h_0)*, what is the field* $\widehat{h}$ *for the corresponding open circle, in terms of* a*,* b*,* ϵ*, and the original coordinates?* (You may use the relation between $\widehat{h}$ and h_0 to check your answer for part (a).)

The magnetization $M(t,h)$ is observed to rescale under this same coarse-graining operation to $M' = (1+c\epsilon)\, M$, so $M\left((1+a\epsilon)\, t, (1+b\epsilon)\, h\right) = (1+c\epsilon)\, M(t,h)$.

(c) *Suppose* $M(t,h)$ *is known at* (t_0, h_0)*, one of the solid dots. Give a formula for* $M(2t_0, \widehat{h})$ *at the corresponding open circle, in terms of* $M(t_0, h_0)$*, the original coordinates, a, b, c, and* ϵ*.* (Hint: Again, rescale n times.) *Substitute your formula for* $\widehat{h}$ *into the formula, and solve for* $M(t_0, h_0)$*.*

You have now basically derived the key result of the renormalization group; the magnetization curve at t_0 can be found from the magnetization curve at $2t_0$. In Section 12.2, we shall coarse-grain not to $t = 2t_0$, but to $t = 1$. We shall see that the magnetization everywhere can be predicted from the magnetization where the invariant curve crosses $t = 1$.

(d) *There was nothing about the factor of two in our shift in temperature that was special. Substitute* $1/t_0$ *for 2 in your formula from part (c). Show that* $M(t,h) = t^{\beta}\mathcal{M}(h/t^{\beta\delta})$ *(the standard scaling form for the magnetization in the Ising model). What are* β *and* δ *in terms of a, b, and c? How is* $\mathcal{M}$ *related to* $M(t,h)$ *where the curve crosses* $t = 1$*?*

Note that we have succeeded in writing $M(t,h)$ in the two-dimensional plane in terms of its value $M(1,h)$ along the line $t = 1$. A property depending on n variables near a critical point has a singular part that can be written as a power law in one variable times a scaling function $\mathcal{M}$ of $n-1$ variables. What is more, the power law and the scaling function is universal – shared between all systems that can flow to the same renormalization-group fixed point.

12.11 **The renormalization group and the central limit theorem: long.** (Mathematics)④

In this exercise, we will develop a renormalization group in *function space* to derive the central limit theorem [1]. We will be using maps (like our renormalization transformation T) that take a function ρ of x into another function of x; we will write $T[\rho]$ as the new function, and $T[\rho](x)$ as the function evaluated at x. We will also make use of the Fourier transform (eqn A.6)

$$\mathcal{F}[\rho](k) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} \rho(x)\,\mathrm{d}x; \tag{3}$$

$\mathcal{F}$ maps functions of x into functions of k. When convenient, we will also use the tilde notation: $\widetilde{\rho} = \mathcal{F}[\rho]$, so for example (eqn A.7)

$$\rho(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\widetilde{\rho}(k)\,\mathrm{d}k. \tag{4}$$

The central limit theorem states that the sum of many independent random variables tends to a Gaussian, whatever the original distribution might have looked like. That is, the Gaussian distribution is the *fixed-point function* for large sums. When summing many random numbers, the details of the distributions of the individual random variables becomes unimportant; simple behavior emerges. We will study this using the renormalization group, giving an example where we can explicitly implement the coarse-graining transformation. Here our system space is the space of probability distributions $\rho(x)$. There are four steps in the procedure.

[1] *Coarse-grain.* Remove some fraction (usually half) of the degrees of freedom. Here, we will add pairs of random variables; the probability distribution for sums of N independent random variables of distribution f is the same as the distribution for sums of $N/2$ random variables of distribution $f * f$, where $*$ denotes convolution.

(a) *Argue that if $\rho(x)$ is the probability that a random variable has value x, then the probability distribution of the sum of two random variables drawn from this distribution is the convolution*

$$C[\rho](x) = (\rho * \rho)(x) = \int_{-\infty}^{\infty} \rho(x-y)\rho(y)\,\mathrm{d}y. \tag{5}$$

Remember (eqn A.23) the Fourier transform of the convolution is the product of the Fourier transforms, so

$$\mathcal{F}[C[\rho]](k) = \left(\widetilde{\rho}(k)\right)^2. \tag{6}$$

[2] *Rescale.* The behavior at larger lengths will typically be similar to that of smaller lengths, but some of the constants will shift (or *renormalize*). Here the mean and width of the distributions will increase as we coarse-grain. We confine our main attention to distributions of zero mean. Remember that the width (standard deviation) of the sum of two random variables drawn from ρ will be $\sqrt{2}$ times the width of one variable drawn from ρ, and that the overall height will have to shrink by $\sqrt{2}$ to stay normalized. We define a rescaling operator $S_{\sqrt{2}}$ which reverses this spreading of the probability distribution:

$$S_{\sqrt{2}}[\rho](x) = \sqrt{2}\rho(\sqrt{2}x). \tag{7}$$

(b) *Show that if ρ is normalized (integrates to one), so is $S_{\sqrt{2}}[\rho]$. Show that the Fourier transform is*

$$\mathcal{F}[S_{\sqrt{2}}[\rho]](k) = \widetilde{\rho}(k/\sqrt{2}). \tag{8}$$

Our renormalization-group transformation is the composition of these two operations,

$$\begin{aligned} T[\rho](x) &= S_{\sqrt{2}}[C[\rho]](x) \\ &= \sqrt{2}\int_{-\infty}^{\infty} \rho(\sqrt{2}x - y)\rho(y)\,\mathrm{d}y. \end{aligned} \tag{9}$$

Adding two Gaussian random variables (convolving their distributions) and rescaling the width back should give the original Gaussian distribution; the Gaussian should be a *fixed-point.*

(c) *Show that the Gaussian distribution*

$$\rho^*(x) = (1/\sqrt{2\pi}\sigma)\exp(-x^2/2\sigma^2) \tag{10}$$

is indeed a fixed-point in function space under the operation T. You can do this either by direct integration, or by using the known properties of the Gaussian under convolution.

(d) *Use eqns 6 and 8 to show that*

$$\mathcal{F}[T[\rho]](k) = \widetilde{T}[\widetilde{\rho}](k) = \widetilde{\rho}(k/\sqrt{2})^2. \tag{11}$$

Calculate the Fourier transform of the fixed-point $\widetilde{\rho^*}(k)$ (or see Exercise A.4). Using eqn 11, show that $\widetilde{\rho^*}(k)$ is a fixed-point in Fourier space under our coarse-graining operator $\widetilde{T}$.[1]

These properties of T and ρ^* should allow you to do most of the rest of the exercise without any messy integrals.

The central limit theorem tells us that sums of random variables have probability distributions that approach Gaussians. In our renormalization-group framework, to prove this we might try to show that our Gaussian fixed-point is *attracting*: that all nearby probability distributions flow under iterations of T to ρ^*.

[3] *Linearize about the fixed point.* Consider a function near the fixed point: $\rho(x) = \rho^*(x) + \epsilon f(x)$. In Fourier space, $\widetilde{\rho}(k) = \widetilde{\rho^*}(k) + \epsilon \widetilde{f}(k)$. We want to find the eigenvalues λ_n and eigenfunctions f_n of the derivative of the mapping T. That is, they must satisfy

$$\begin{aligned} T[\rho^* + \epsilon f_n] &= \rho^* + \lambda_n \epsilon f_n + O(\epsilon^2), \\ \widetilde{T}[\widetilde{\rho^*} + \epsilon \widetilde{f_n}] &= \widetilde{\rho^*} + \lambda_n \epsilon \widetilde{f_n} + O(\epsilon^2). \end{aligned} \tag{12}$$

(e) *Show using eqns 11 and 12 that the transforms of the eigenfunctions satisfy*

$$\widetilde{f_n}(k) = (2/\lambda_n)\widetilde{\rho^*}(k/\sqrt{2})\widetilde{f_n}(k/\sqrt{2}). \tag{13}$$

[4] *Find the eigenvalues and calculate the universal critical exponents.*

(f) *Show that*

$$\widetilde{f_n}(k) = (\mathrm{i}k)^n \widetilde{\rho^*}(k) \tag{14}$$

is the Fourier transform of an eigenfunction (i.e., that it satisfies eqn 13.) What is the eigenvalue λ_n?

Our fixed-point actually does not attract all distributions near it. The directions with eigenvalues greater than one are called *relevant*; they are dangerous, corresponding to deviations from our fixed-point that grow under coarse-graining. The directions with eigenvalues equal to one are called *marginal*; they do not get smaller (to linear order) and are thus also potentially dangerous. When you find relevant and marginal operators, you always need to understand each of them on physical grounds.

[1] To be explicit, the operator $\widetilde{T} = \mathcal{F} \circ T \circ \mathcal{F}^{-1}$ is a renormalization-group transformation that maps Fourier space into itself.

(g) *The eigenfunction* $f_0(x)$ *with the biggest eigenvalue corresponds to an unphysical perturbation; why?* (Hint: Probability distributions must be normalized to one.) *The next two eigenfunctions* f_1 *and* f_2 *have important physical interpretations. Show that* $\rho^* + \epsilon f_1$ *to is equivalent to a shift in the mean of* ρ*, and* $\rho^* + \epsilon f_2$ *is a shift in the standard deviation* σ *of* ρ^**, to linear order in the shifts.*

In this case, the relevant perturbations do not take us to qualitatively new phases—just to other Gaussians with different means and variances. All other eigenfunctions should have eigenvalues λ_n less than one. This means that a perturbation in that direction will shrink under the renormalization-group transformation:

$$T^N(\rho^* + \epsilon f_n) - \rho^* \sim \lambda_n^N \epsilon f_n. \tag{15}$$

Corrections to scaling and coin flips. Does anything really new come from all this analysis? One nice thing that comes out is the *leading corrections to scaling*. The fixed-point of the renormalization group explains the Gaussian shape of the distribution of N coin flips in the limit $N \to \infty$, but the linearization about the fixed-point gives a systematic understanding of the corrections to the Gaussian distribution for large but not infinite N.

Usually, the largest eigenvalues are the ones which dominate. In our problem, consider adding a small perturbation to the fixed-point f^* along the two leading irrelevant directions f_3 and f_4:

$$\rho(x) = \rho^*(x) + \epsilon_3 f_3(x) + \epsilon_4 f_4(x). \tag{16}$$

These two eigenfunctions can be inverse-transformed from their k-space form (eqn 14):

$$\begin{aligned} f_3(x) &\propto \rho^*(x)(3x/\sigma - x^3/\sigma^3), \\ f_4(x) &\propto \rho^*(x)(3 - 6x^2/\sigma^2 + x^4/\sigma^4). \end{aligned} \tag{17}$$

What happens to these perturbations under multiple applications of our renormalization-group transformation T? After ℓ applications (corresponding to adding together 2^ℓ of our random variables), the new distribution should be given by

$$T^\ell(\rho)(x) \sim \rho^*(x) + \lambda_3^\ell \epsilon_3 f_3(x) + \lambda_4^\ell \epsilon_4 f_4(x). \tag{18}$$

Since $1 > \lambda_3 > \lambda_4 \dots$, the leading correction should be dominated by the perturbation with the largest eigenvalue.

(h) *Plot the difference between the binomial distribution giving the probability of* m *heads in* N *coin flips, and a Gaussian of the same mean and width, for* $N = 10$ *and* $N = 20$. (The Gaussian has mean of $N/2$ and standard deviation $\sqrt{N}/2$, as you can extrapolate from the case $N = 1$.) *Does it approach one of the eigenfunctions* f_3 *or* f_4 *(eqns 17)?*

(i) *Why did a perturbation along* $f_3(x)$ *not dominate the asymptotics? What symmetry forced* $\epsilon_3 = 0$*? Should flips of a biased coin break this symmetry?*

Using the renormalization group to demonstrate the central limit theorem might not be the most efficient route to the theorem, but it provides quantitative insights into how and why the probability distributions approach the asymptotic Gaussian form.

References

[1] Chayes, J. T., Chayes, L., Sethna, J. P., and Thouless, D. J. (1986). A mean-field spin glass with short-range interactions. *Communications in Mathematical Physics*, **106**, 41.