Physics 7653: Statistical Physics
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In Class Exercises
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12.28 **Singular corrections to scaling.** ③

The renormalization group says that the number of relevant directions at the fixed point in system space is the number of parameters we need to tune to see a critical point, and that the critical exponents depend on the eigenvalues of these relevant directions. Do the irrelevant directions matter?

Let the Ising model in zero field be described by flow equations

$$\mathrm{d}t_\ell/\mathrm{d}\ell = t_\ell/\nu, \qquad \mathrm{d}u_\ell/\mathrm{d}\ell = -y u_\ell \tag{1}$$

where t_ℓ describes the renormalization of the reduced temperature $t = (T_c - T)/T_c$ after a coarse-graining by a factor $b = \exp(\ell)$, and u and u_ℓ represent a slowly-decaying irrelevant perturbation under the renormalization group. In Fig. 12.8, one may view t as the expanding eigendirection running roughly horizontally, and u as the contracting, irrelevant coordinate running roughly vertically. Thus our model starts with a value u_0 associated to the distance in system space between our critical point R_c and the RG fixed point S^* along the irrelevant coordinate.

(a) *What is the invariant combination $z = ut^\omega$ that stays constant under the renormalization group? What is ω in terms of the eigenvalues $-y$ and $1/\nu$?*

Properties near critical points have universal power law singularities, but the corrections to these power laws also have universal properties predicted by the renormalization group. These come in two types – *analytic* corrections to scaling and *singular* corrections to scaling.

Let us consider corrections to the susceptibility. In analogy with other systems we have studied, we would expect that the susceptibility

$$\chi(t, u) = t^{-\gamma} X(z) \tag{2}$$

with $X(z)$ a universal function of the invariant combination you found in part (a). (We shall derive this scaling form in Exercise 12.29.) As a function of t, $\chi(t, u)$ has singularities at small t. But we expect properties to be analytic as we vary u, since the irrelevant direction is not being tuned to a special value, so we expect that a Taylor series of $\chi(t, u)$ in powers of u should make sense. Since $z \propto u$, we thus expect that $X(z)$ will be an analytic function of z for small z.[1]

[1]Had we used a scaling variable $Z = tu^{1/\omega}$, for example, we would not have expected the corresponding scaling function to be analytic in small Z.

(b) *Show that for small t, your z from part (a) goes to zero. Taylor expand $X(z)$. What corrections do you predict for the susceptibility from the first and second-order terms in the series?* These are the singular corrections to scaling due to the irrelevant perturbation u.

An Ising magnet on a sample holder is loaded into a magnetometer, and the susceptibility is measured[2] at zero external field as a function of reduced temperature $t = (T - T_c)/T_c$. It is found to be well approximated by

$$\begin{aligned}\chi(T) = At^{-1.24} + Bt^{-0.83} + Ct^{-0.42}\\ + D + Et + \ldots\end{aligned} \tag{3}$$

You may ignore any errors due to the magnetometer.

(c) *The exponent $\omega \approx 0.407$ for the 3D Ising universality class, and $\gamma \approx 1.237$. Which terms are explained as singular corrections to scaling?*

(d) *Can you provide a physical interpretation for the terms in eqn 3 that are not explained by singular corrections to scaling? For example, how do we expect the susceptibility of the sample holder to depend on temperature?* These are examples of *analytic* corrections to scaling.

One must note that it is normally completely infeasible in an experiment or simulation to measure quantities with sufficiently accuracy to identify so many simultaneous corrections to scaling.

12.29 **Singular corrections to scaling and the renormalization group.** ③

In this exercise, we derive the form of the scaling function (eqn 2) for the effects of *irrelevant* operators on the properties of systems near critical points (see Exercise 12.28). Remember that irrelevant directions shrink under coarse-graining. Let χ be the susceptibility of the Ising model, as a function of the reduced temperature $t = T_c - T$ and some irrelevant operator u:

$$\begin{aligned}d\chi_\ell/d\ell &= -(\gamma/\nu)\chi_\ell,\\ dt_\ell/d\ell &= t_\ell/\nu,\\ du_\ell/d\ell &= -yu_\ell\end{aligned} \tag{4}$$

How do we derive the universal scaling function $X(z)$ from these renormalization group flows? Consider the flows illustrated in Fig. 12.8, except now with a third dimension involving the prediction χ. Consider a point (t_0, u_0, χ_0) in the system space, and the invariant curve defined by $z = u_0 t_0^\omega$ (dashed lines). Our renormalization group allows us to calculate $\chi_\ell(t_\ell, u_\ell)$ along these curves – relating the behavior everywhere near the critical manifold (vertical swath flowing toward S^*) to the properties along the outgoing trajectories, which approach closer and closer to the unstable manifold (the horizontal swath flowing away from S^*.

[2] The accuracy of the quoted exponents is not experimentally realistic.

For example, we can define the universal scaling function $X(z)$ (for positive time t) to be the χ_{ℓ^*} where the flow crosses $t_{\ell^*} = 1$.

(a) *Solve eqns 4 for u_ℓ and t_ℓ. Setting $t_{\ell^*} = 1$, what is u_{ℓ^*} in terms of your invariant combination z?*

So we label each invariant scaling curve by the value of the vertical position u_{ℓ^*} where it crosses $t_{\ell^*} = 1$.

(b) *Solve eqns 4 for $\chi_{\ell^*}(1, u_{\ell^*})$, in terms of z, t_0, and $\chi_0(t_0, u_0)$. Use your solution to solve for the physical behavior $\chi_0(t_0, u_0)$ in terms of t and $X(z)$. Express $X(z)$ in terms of $\chi_{\ell^*}(1, u_{\ell^*})$.* Does your answer agree with the form in eqn 2?

Remember the critical manifold is co-dimension one (or two, if you include temperature and external field), and the unstable manifold is dimension one (or two) – so we get universal predictions for a huge variety of systems, by observing the outgoing trajectories near a narrow tube or surface emitted from the fixed point.