

12.30 Nonlinear flows, analytic corrections, and hyperscaling.¹ ③

We consider the effects of nonlinear terms in renormalization group flows. The Ising model in zero field has one relevant variable (the deviation t of the temperature from T_c). To calculate the specific heat, we shall also consider the flow of the free energy per spin f under the renormalization group. Instead of a discrete coarse-graining by a factor b , here we use a continuous coarse-graining measured by ℓ . One can think of one coarse-graining step by $b = (1 + \epsilon)$ as incrementing $\ell \rightarrow \ell + \epsilon$; equivalently, coarse-graining to ℓ changes length scales by $\exp(\ell)$.

Consider the particular flow equations²

$$df_\ell/d\ell = Df_\ell - at_\ell^2 \quad dt_\ell/d\ell = t_\ell/\nu, \quad (1)$$

where D is the dimension of space and at_ℓ^2 is a nonlinear term that will be important in two dimensions.

Notice there are several free energies here. We shall call the free energy per spin of the actual system either f or f_0 , and the temperature of the actual system either t or t_0 . The coarse-grained free energy and temperature are f_ℓ and t_ℓ , after being coarse-grained by a factor $\exp(\ell)$. (Hence at $\ell = 0$ we have not yet coarse-grained, so $f_0 = f$ and $t_0 = t$.) Notice here that the free energy of our system is the *initial condition* $f_0(t_0)$ of this differential equation, and $f_\ell = f(\ell)$ and $t_\ell = t(\ell)$ are the renormalization group flows of the two variables. To derive the scaling behavior, we shall coarse-grain to ℓ^* where $t_{\ell^*} = 1$, at which point the coarse-grained free energy is f_{ℓ^*} .

Let us start with the linear case $a = 0$.

(a) *Solve for f_ℓ and t_ℓ for $a = 0$. Setting $t_{\ell^*} = 1$, solve for f_0 in terms of f_{ℓ^*} , t_0 , D , and ν . Solve for the specific heat per spin $c = T\partial^2 f/\partial T^2$, where $t = t_0 = (T - T_c)/T_c$ and $f = f_0$. (Hint: Use the chain rule $\partial f/\partial T = (\partial f/\partial t)(dt/dT)$.) Show that the specific heat near T_c has a power-law singularity $c \propto t^{-\alpha}$, with $\alpha = 2 - D\nu$. (For example, in $D = 3$, $\nu \approx 0.63$, so $\alpha = 2 - D\nu \sim 0.11$; the specific heat diverges at T_c .) Writing*

$$c = t^{-\alpha}(c_0 + c_1 t + c_2 t^2 + \dots), \quad (2)$$

¹This exercise developed in collaboration with Colin Clement

²Note that these are total derivatives. So the first equation tells us the total change in f after coarsening by a factor $1 + d\ell$. $f(t)$ then will coarse grain to $f_\ell(t_\ell)$ without needing to worry about the chain rule $df(t)/d\ell = \partial f/\partial \ell + \partial f/\partial t dt/d\ell$.

what is the first correction c_1 to the specific heat near $t = 0$ in the absence of the nonlinear term?

Why is the linear term in the the free energy flow equal to the dimension, $df/d\ell = Df + \dots$, where all other terms are hard-to-compute critical exponents? There is no completely general answer to this question (although there are arguments for specific models). Indeed, other models of disordered systems and glasses, and models above the upper critical dimension, the linear term is not given by the dimension. The relation $2 - \alpha = D\nu$ is called a *hyperscaling* relation (to emphasize they involve the dimension D), and these other models are said to violate hyperscaling.

(b) In the case $a = 0$, show that the singular free energy f contained in a correlated volume ξ^D near the critical temperature becomes independent of the distance to the critical point. (Hint: Look up the critical exponent describing how the correlation length ξ diverges as $t \rightarrow 0$.)

Glassy and disordered systems become extremely sluggish as they are cooled. In at least some cases, this is precisely because the energy barriers needed to continue equilibration diverge as their correlation lengths grow – they are glassy because their RG flows violate the hyperscaling relation.

So much for the power law singularity – what about the correction term c_1 in part (a)? It is an *analytic correction to scaling*.³ Here it is *subdominant* – near the critical temperature where $t \rightarrow 0$, it is less singular than the leading term. One expects in a real physical system that the microscopic bond free energy J between spins will be some analytic function of temperature, and the physical free energy and specific heat will be multiplied by J . Expanding J in a Taylor series about $t = 0$ would also give us terms like those in eqn 2.

Does the introduction of the higher-order nonlinear term at_ℓ^2 in eqn 1 change the behavior in an important way? Rather than exercising your expertise in analytic solutions of nonlinear differential equations, eqn 3 provides not f_ℓ and t_ℓ as functions of ℓ , but the relation between the two:

$$f_\ell(t_\ell) = f_0 \left(\frac{t_\ell}{t_0} \right)^{D\nu} - \frac{a\nu t_\ell^2}{(2 - D\nu)} (1 - (t_\ell/t_0)^{-(2-D\nu)}). \quad (3)$$

(c) Show that $f_\ell(t_\ell)$ in eqn 3 satisfies the differential equation given by eqn 1, using

$$\frac{df_\ell}{dt_\ell} = \frac{df_\ell}{d\ell} \bigg/ \frac{dt_\ell}{d\ell} \quad (4)$$

³These are distinct from *singular* corrections to scaling that arise, for example, from irrelevant terms under the renormalization group, that would produce subdominant corrections to c with powers $t^{-\alpha+\Delta}$ where Δ is not an integer, and is bigger than zero (hence subdominant).

Show that it has the correct initial conditions at $\ell = 0$. What is f_{ℓ^*} at ℓ^* , where $t_\ell = 1$? Show your method.

Again, it is important to remember that $f_\ell(t_\ell)$ is not the free energy as a function of temperature – it is the coarse-grained free energy as a function of the coarse-grained temperature of a system starting at a free energy f_0 at a temperature t_0 . It is $f_0(t_0)$ that we want to know. Since here we have only one relevant variable (in zero field), all the flows lead to the same⁴ final point $f_{\ell^*}(t_{\ell^*} = 1)$

(d) Solve for f_0 in terms of f_{ℓ^*} and t_0 . Solve for the specific heat $c = T\partial^2 f/\partial T^2$, where $t = (T - T_c)/T_c$. Show that it can be written in the form

$$c = c_{+\text{analytic}}(t) + t^{-\alpha}c_{*\text{analytic}}(t) \quad (5)$$

where the additive correction $c_{+\text{analytic}}(t)$ and the multiplicative correction $c_{*\text{analytic}}(t)$ have a simple Taylor series about $t = 0$. Write these two corrections, in terms of f_{ℓ^*} , T_c , a , ν , and D .

Here the nonlinear term a gives us not only a smooth multiplicative term in the specific heat, but also a smooth additive background. This clearly is also expected in a real physical system – other degrees of freedom uncoupled to the magnetization, or even the box holding the magnet, will contribute a specific heat that is analytic near T_c .

12.31 Beyond power laws: Nonlinear flows and logarithms in the 2D Ising model.⁵ ③

The two dimensional Ising model has a logarithmic singularity in the specific heat. The exact result shows that the specific heat per spin is

$$\begin{aligned} c(T) &= k_B \frac{2}{\pi} \left(\frac{2J}{k_B T_c} \right)^2 \left[-\log(1 - T/T_c) \right. \\ &\quad \left. + \log(k_B T_c / (2J) - (1 + \pi/4)) \right] \\ &= -\frac{8}{\pi k_B T_c^2} \log \left(\frac{t}{\frac{1}{2} k_B T_c \exp(-(1 + \pi/4))} \right) \\ &= -c_0 \log \left(\frac{t}{\tau} \right) \end{aligned} \quad (6)$$

where $t = (T - T_c)/T_c$ and we set $J = 1$. (Remember $\log(t)$ is negative for small t .) Linearized flows around the renormalization group fixed point predict $c \sim t^{-\alpha}$, and when $\alpha \rightarrow 0$ one often observes logarithmic corrections. But such corrections are not predicted by the linearized flows. The key nonlinear term is the term at_ℓ^2 of eqn 1 in Exercise 12.30.

⁴Remember for systems with more than one variable (say t and h), the free energy depends on the invariant curve departing from the fixed point (labeled, say, by $h/t^{\beta\delta} = h_{\ell^*}(t_{\ell^*} = 1)$). We solve for $f_0(t_0, h_0)$ in terms of $f_{\ell^*}(1, h_{\ell^*})$ just as we do in part (b).

⁵This exercise developed in collaboration with Colin Clement

(a) *Is the solution for $f_\ell(t_\ell)$ in eqn 3 useful in $D = 2$? Why or why not? (Hint: The exponent $\nu = 1$ for the two-dimensional Ising model.)*

Again, we provide the solution of the nonlinear RG eqns 1 for $D = 2$

$$f_\ell(t_\ell) = f_0(t_\ell/t_0)^2 - at_\ell^2 \log(t_\ell/t_0). \quad (7)$$

(b) *Show that $f_\ell(t_\ell)$ in eqn 7 satisfies the differential equation given by eqn 1, with the correct initial conditions. Solve for f_0 in terms of f_{ℓ^*} and t_0 , where $t_{\ell^*} = 1$. Solve for the specific heat $c = T\partial^2 f/\partial T^2$, where $t = t_0 = (T - T_c)/T_c$ and $f = f_0$. (Remember the chain rule: $\partial f/\partial T = (\partial f/\partial t)(dt/dT)$.) Does it agree asymptotically with the exact result in eqn 6? What are c_0 and τ , in terms of a and f_{ℓ^*} ?*

Thus for the 3D Ising model (Exercise 12.30), nonlinear terms in the renormalization-group flow equations give only analytic corrections to scaling, where in the 2D Ising model they introduce logarithms in the specific heat. Normal form theory (see Exercise 12.4) can be used to determine when one may safely linearize these equations, and to organize the other critical points into universality *families* [1].

References

- [1] Raju, A., Clement, C. B., Hayden, L. X., Kent-Dobias, J. P., Liarte, D. B., Rocklin, D. Z., and Sethna, J. P. (2017). Renormalization group and normal form theory. (*submitted*).