Advanced topics for group projects
Statistical Physics 653, Fall 2010, James P. Sethna
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Below find a brief discussion of applications of the Renormalization Group to varied fields of physics and science. I'm also happy to entertain other suggested topics.

(0) ϵ **expansion and** $1/N$ **expansion methods.** Fluctuations become less important in higher dimensions. Above an 'upper critical dimension' (usually four for non-disordered systems) continuous phase transitions can be described by mean-field theory, ignoring the fluctuations. Wilson's original renormalization-group calculation was a perturbation about mean-field theory, using *dimension* as a small parameter (the $4-\epsilon$ expansion). Cardy uses the "operator product expansion" (OPE) to implement this ϵ expansion, rather different from the Feynman-diagram methods used by most other review articles and texts.

In 2009, Zach Lamberty and Darryl Ngai covered this topic. We could also try to develop an exercise that implements the ϵ expansion (or perhaps a $1/N$ expansion) in a context where the algebra isn't too messy and the problem is accessible and interesting.

(1) **Kosterlitz-Thouless and topological phases.** Fluctuations become more important in low dimensions. Indeed, below the 'lower critical dimension' for a given system the fluctuations drive the critical temperature to zero. At the lower critical dimension, the phase transition often has unusual scaling – not power laws, but exponentials in some quantities and sometimes jumps in others. The freezing of crystals and superfluids in two dimensions, along with fluctuations in the shapes of (three-dimensional) crystal surfaces, are described by the Kosterlitz-Thouless transition. Much of the beautiful physics of this transition (and especially the related 2D melting of Halperin Nelson and Young) is hidden by the elastic theory and other details needed before coarse-graining. Basically the same renormalization-group ideas can be found with fewer complications in the one-dimensional Ising model with inverse-square interactions.

In 2008 Ben Hunt presented theory and (Cornell) experiments on superfluid films, and Turan Birol and Wee Don Teo worked out and presented the renormalization-group treatment of the inverse-square Ising model. (a) We could try to develop an exercise that illustrates the features of topological order – phase transitions without broken symmetries. (b) There are deep connections between these classical systems and the quantum physics of point defects coupled to Fermi gases: we could work out and present Leggett's theory of *macroscopic quantum tunneling*, and perhaps the Anderson/Yuval/Wilson theory of the Kondo effect, or other similar applications (quantum tunneling of atoms off of STM tips?)

(2) **Disordered and glassy systems.** The study of dirt (random impurities and defects frozen into the system) and its effects both on phases and phase transitions is a rich and controversial field of statistical mechanics. Dirt can act to dramatically slow the dynamics (long-time tails, glassy behavior), act to freeze the dynamics (spin-glass transitions), or lead to qualitatively new phenomena (localization of electron wave-functions). The effects of randomness often can be separated into 'typical' (Imry-Ma) and 'extreme' (Griffiths) fluctuations.

In 2008, Robin Baur, YJ Chen, Vikram Gadagkar, and Joe Chen did an in-depth treatment of the equilibrium behavior of the random-field Ising model (RFIM), which illustrates *dangerous irrelevant variables* and leads, by fairly simple arguments, to diverging barriers to relaxation. Vikram also touched on the Harris criterion, which governs whether "random T_c" disorder is relevant or irrelevant at a phase transition. The RFIM is also a morality tale, with dueling experimental and theoretical views, abuses, politics, and noble behavior, with a resolution that makes everyone correct and yet both sides look sheepish. (a) We could move on to describe one of the many other disordered systems – perhaps localization and the metal-insulator transition. (b) We could develop an exercise on diverging barriers, perhaps based on the next-neighbor Ising model ("Prediction of Logarithmic Growth in a Quenched Ising Model" Joel D. Shore, and James P. Sethna, *Phys. Rev. B* **43**, 3782 (1991)).

(3) **Hydrodynamics, fluctuations, flocking, and all that.** New physics often emerges on long length and time scales, even in the absence of phase transitions. Hydrodynamics – originally the study of the flow of fluids – has now been generalized to encompass the multiplicity of systems where complex microscopic interactions lead to simple continuum laws. The general approach is elegant: write down the most general theory allowed by symmetry, expanding in powers of gradients and order parameters, and then keep the most important terms. In the absence of noise terms, this procedure generalizes the familiar continuum limits we take in deriving the wave equation and elastic theory. With noise, things get trickier, and renormalization-group ideas come in.

In 2008,, Gordon Berman discussed *flocking* – the motion of flocks of animals. He noted in the end that the renormalization-group theory of the "flocking transition" (where all the animals coordinate just enough to end up migrating in the same direction) seems still confusing – perhaps abrupt and not continuous. The flocking phase in two dimensions, though, is remarkable: the existence of long-range order in 2D flocking would seem to violate the Mermin-Wagner theorem . . . In 2010, Jesse Silverberg, Matt Bierbaum, and Adam Holmes did experiments and simulations for human flocking behavior at hard rock concerts, studying the transition between mosh pits and circle pits. (a) There are theories of fluctuating rods and theories of traffic jams that we could present, or even develop into exercises. (b) I have an exercise on the wave equation that currently ignores noise, and we should see if we can show that noise is irrelevant there.

(4) **Fermi liquid theory and the renormalization group.** Why are metals described so well in terms of non-interacting electrons filling a Fermi sea? The traditional theory – Green's functions and quasiparticles, Fermi liquid theory, and diagrams – is powerful and useful, but doesn't tie in well with modern statistical mechanics and field theory. A wonderful new approach treats the Fermi sea as a renormalization-group fixed point, where the coarse-graining removes the high-energy electrons and holes, incorporating their screening effects in renormalized interaction strengths between the quasiparticles. This renormalization-group is unusual, in that there are an infinite number of *marginal* directions (neither relevant nor irrelevant). These marginal directions are quadratically unstable to attractive interactions, which are responsible for the transformation of metals at low temperatures into superconductors, spin-density waves and charge-density waves. The transition temperature is neatly explained using the renormalization group flows.

In 2008, Brian Daniels, Duane Loh, Mark Transtrum, and Johannes Lischner presented a brief summary of the traditional Green's function / quasiparticle approach to Fermi liquid theory, an intuitive introduction to the renormalization-group approach, a discussion of how to use quadratically unstable modes to understand the BCS formula for the transition temperature of superconductors, and a whirlwind introduction to the technicalities of coarse-graining the interacting Fermi liquid. In 2010, Mihir Khadilkar, Shivam Ghosh, Kartik Ayyer, and Kyungmin Lee discussed both these theories and quantum phase transitions. (a) Duncan Haldane some twenty years ago gave some talks about how the Fermi surface was the 'order parameter', and I think that the quasiparticles somehow were related (via bosonization?) to Goldstone modes. I don't believe I ever heard these talks except second-hand, and I don't think these ideas were ever published, but we could think deeply and see if we can make any sense of them.

(5) **Quantum phase transitions.** Quantum mechanics is usually not needed to describe the exponents and asymptotic behavior near finite-temperature phase transitions (quantum fluctuations are *irrelevant* perturbations at finite-temperature critical points). However, there are phase transitions at zero temperature (like the metal/insulator, superconductor/insulator, and quantum Hall plateau transitions). More important, a transition at zero temperature can dominate a large region of the finite-temperature phase diagram. The behavior right on the critical line might be classical, but (oddly) the quantum effects can dominate farther away. Much recent activity, both in high-T_c systems and in other quantum magnets, has been interpreted in terms of quantum fluctuations due to a zero-temperature fixed point.

In 2008, Yang Xie gave a broad introduction to the experiments on high-T_c superconductivity and the evidence suggesting an underlying quantum critical point. Hitesh Changlani, JeeHye Lee, Stefan Natu, and Ben Heidenreich, Josh Berger introduced the connection between d-dimensional quantum statistical mechanics and $d+1$-dimensional classical statistical mechanics and explained the scaling behavior of finite-temperature quantum critical phenomena using universal crossover functions. In 2010, Mihir Khadilkar, Shivam Ghosh, Kartik Ayyer, and Kyungmin Lee discussed both these theories and Fermi liquid theory. (a) There is a somewhat separate discussion of critical phenomena in quantum systems coupled to heat baths which goes under the name of macroscopic quantum tunneling. How these ideas are related are unclear, but might be worth pursuing.

(6) **Conformal field theory.** At a critical point, the effects of the lattice disappear. An Ising model on a square lattice becomes both translation and rotation invariant on long length scales. The symmetry of the fixed point is larger than the symmetry of the original Hamiltonian: there is an *emergent* rotational symmetry. In addition, another new symmetry arises, incorporating changes of scale – a system at its critical point appears self-similar. One can argue that combining these symmetries, for systems with short-range interactions, naturally leads to a larger invariance under *conformal transformations* – transformations that are rotations and dialations that vary in space. In two dimensions, conformal invariance is a huge symmetry group: basically one transformation for every analytic function on the com-

plex plane. In the study of string theory, it was discovered that the possible critical points in two dimensions can be categorized by using this huge symmetry group. This is fundamentally our best explanation of why so many two-dimensional critical points have exact solutions (with simple rational values for critical exponents), while essentially no higher-dimensional models are exactly solvable or have simple exponents (except when mean-field theory applies).

Stochastic Loewner evolution. As a related technique, it has recently been discovered by mathematicians that the *interfaces* at two-dimensional critical points (between up and down in the Ising model, around the percolation cluster, etc.) are all generated by a generalization of a random walk, called stochastic Loewner evolution (SLE). Different models are described by different values of a constant κ. This appears to be both cool and mathematically powerful.

In 2008, Justin Vines, Phil Kidd, Ben Machta and Srivatsen Chakram gave a cool presentation of conformal invariance and SLE, with software distorting square simulations of the Ising model into weird shapes, software generating the fractal boundaries for different κ, discussions of how to classify all 2D theories and calculate exponents, and discussions of how these methods can be used to calculate correlation functions. (a) Writing up these ideas into a chapter would seem a great idea. (b) Transforming their software into a general-purpose conformal transformation tool would be great. I envision an exercise in every stat mech class where you distort an Ising model / percolation model / etc. by any analytic function you care to type in, and see that at T_c that you can't tell where it's been stretched. I have Justin's notes and PowerPoint, and a video DVD of one of the presentations.

(7) **Recent developments in the theory of glasses.** Glasses are disordered like liquids, but rigid like crystals. Unlike spin glasses and other well-studied disordered systems, the disorder in glasses freezes in as the glass becomes rigid. Fundamentally, there is no consensus about why glasses become so viscous over such a small range of temperatures. While I still believe it is possible and likely that glasses represent some kind of avoided thermodynamic phase transition, current interesting theories describe it as a purely dynamical phenomena. Jamming theories due to Andrea Liu and collaborators (former Cornell grad), new analysis of kinetically constrained models by Daniel Fisher and collaborators (Cornell child and undergrad), mode-coupling theories by Jean-Philippe Bouchaud and collaborators, and now some wonderful new curved-space simulations of Gilles Tarjus and collabo-

rators make this an exciting new area of research.

(8) **Networks.** There has been a flurry of high-profile papers analyzing the structures of networks in the real world, and trying to duplicate the statistical structures and qualitative features using statistical mechanical models. Small world networks ("six degrees of separation"), random networks, scale-free networks (with power-law distributions for the number of neighbor links for a node), and grown networks are all topics that have scaling and renormalization-group implications, as well as implications for the spread of disease and the vulnerability of the Internet.

In 2009, Igor Segota, Tom Payne, and Alex Alemi discussed phase transitions on networks, and disease propagation (with an entertaining simulation of zombie dynamics).

(9) **Chaos and turbulence.** The renormalization group has been central to our understanding of the onset of chaos. The period doubling route to chaos, the quasiperiodic route to chaos that I studied as a post-doc, and the breakdown of the last KAM torus that governs particle accelerators would make a great topic for a class presentation. There are also some exciting new statistical mechanics developments in the scaling behavior of fully developed turbulence (I remember something about conformal invariance or string theory offshoots)?

(10) **Depinning transitions.** Earthquakes, magnetic Barkhausen noise, plastic deformation of crystals, superconductor breakdown, charge-density wave conduction, raindrops running down windshields – all are *depinning* transitions, with an external force fighting against disorder, leading to a jerky evolution. A rather unified picture has emerged, that would lend itself nicely to a coherent presentation.

(11) **Surface growth.** The deposition of matter onto crystalline surfaces (crystal growth from the melt, molecular beam epitaxy, diffusion limited aggregation (DLA)) and the inverse problem of surface etching have a complex, nonequilibrium dynamics that involves many length and time scales. Faceting, deposition noise, 'uphill' currents from Ehrlich-Schwoebel barriers, and shadowing effects lead to a rich theory. Scaling theories of island growth and aggregation, KPZ and other statistical models, and the subtle multifractal behavior of DLA – all are fascinating problems where renormalization group and scaling are involved.

(12) **Spin glasses, neuroscience, and NP complete problems.** Spin glasses at low temperatures have no long-range order in space, but do have long-range order in time: the spins are frozen into a disordered configura-

tion. Here there are two distinct theoretical threads. Mean-field theories of spin glasses (with infinite-range interactions, representing the behavior in spaces of large dimension) have a wonderfully rich structure. There is a huge hierarchy of ground states (with an *ultrametric* topology), a diverging nonlinear susceptibility, and exotic calculational tools (replica theory and the cavity method) to solve for properties. Cluster models of spin glasses, directly applicable to two and three-dimensional spin glasses, propose or posit only two distinct ground states but study a variety of metastable states formed by flipping clusters of spins with competing interactions. While one could argue that the cluster models may be closer to real spin glasses, the mean-field models have found surprising applications, first in neuroscience (where each neuron is connected to a huge number of neighbors) and now in computational complexity. Indeed, the most successful algorithm known for solving one class of NP complete problems (the most challenging class of computational questions – including the traveling salesman and map coloring problems) is based on the cavity method designed to study spin glasses. Little of this directly involves the renormalization group, but scaling and critical phenomena ideas pervade the subject.

In 2009, Elijah Bogart, Poornima Padmanabhan and Hitesh Changlani discussed NP-complete problems in computer science, and their relations to the random energy model and the number partitioning problem.

(13) **Wetting.** A drop of liquid on a solid surface forms a bead – the liquid, solid, and vapor coexistence at the line of intersection. The contact angles depend on the three surface tensions (liquid-solid, liquid-vapor, and solid-vapor), minimizing the total free energy. If the liquid and solid like one another, the liquid drop will become thin and the contact angle large. At *complete wetting*, the minimum free energy spreads the drop out as a monolayer on the surface. This often happens if the liquid and vapor (or two liquid phases) are near a critical point where they become indistinguishable (and where the interfacial energy cost for the liquid-vapor cap of the droplet goes to zero). Complete wetting is thus often a precursor surface phase transition; Ben Widom and Carl Franck here studied it in great detail theoretically and experimentally earlier in their careers.

(14) **Equilibrium crystal shapes.** Most faceted crystals (like diamonds) are not in equilibrium – the crystals cleave along low energy surfaces, but the atoms have not had time to rearrange between different facets to minimize their energy. The study of crystals that have reached this shape evolution, though, is fascinating. The existence of facets demands a *rough-*

ening transition (in the same universality class as the Kosterlitz-Thouless transition discussed in (1)). The evolution and coarsening of faceted surfaces has a nice scaling theory. The construction of the equilibrium shape from the surface free energy (the Wulff construction) is a multidimensional Legendre transformation. Continuous facet edges have scaling properties determined by two-dimensional (exactly solvable) models. Combining these topics with a renormalization-group theme could be a pretty subject.

In 2009, Kendra Weaver covered the equilibrium crystal shape theory, introducing the Wulff construction. The connection with Kosterlitz-Thouless and roughening, and the connection between facet edges and critical points, remain open problems.

(15) **Asymptotic analysis of partial differential equations.** Asymptotic analysis is an art, not a science. The text of Bender and Orszag provides an introduction to the many sneaky methods one can use to find the asymptotic behavior of various mathematical quantities as parameters get large or small. Nigel Goldenfeld introduced some wonderful new methods for solutions of nonlinear partial differential equations, drawing on ideas and language used in the renormalization-group analysis of phase transitions. He provides an introduction in Chapter 10 of his text, *Lectures on Phase Transitions and the Renormalization Group*, Frontiers in Physics, Addison-Wesley, 1992.

(16) **Bifurcation theory.** Much of the theory of dynamical systems (the time evolution of ordinary differential equations) focuses on *bifurcation theory*, the study of qualitative changes in the dynamics as system parameters evolve. Continuous bifurcations are amazingly analogous to continuous phase transitions, except that they describe systems with only a few active degrees of freedom. Dynamical systems theory lumps together bifurcations into natural families: saddle-node bifurcations, pitchfork bifurcations, Hopf bifurcations, . . . These families each have their *normal form* – analogous to universality classes. A famous theorem shows that any Hopf bifurcation, say, can be transformed into the *normal form* Hopf bifurcation by a continuous change of coordinates. This theorem has its analogy in the *analytic corrections to scaling* needed to understand scaling phenomena at critical points.

In 2010, Alex Moore and Maicol Ochoa Daza did a presentation on these connections, but did not get into analytic corrections to scaling.

(17) **Droplet breakup.** Dripping faucets have hidden universal scaling behavior. As the drops pinch off, the breakup can be wonderfully complicated, with the asymptotic form of the fluid interface forming shapes that

are independent not of the microscopic parameters but (in a reversal of the usual scenario) of the macroscopic way the droplet was formed and exuded. Itai Cohen has been working on this topic since his graduate student years...