Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
Material for Week 4
Exercises due Tuesday Sep 18
Last correction at September 10, 2018, 10:09 am

Pre-class Preparation

Thursday
Read: EOPC, Chapter 12 (Continuous phase transitions).
Do: 12.7 Renormalization-group trajectories.
Submit electronically by 8:30 Thursday morning.

Tuesday
Read: Cardy, Chapter 3 (The Renormalization Group Idea)

Exercises

Start thinking about group projects!

If you did not do this problem in a previous year:
12.32: Conformal invariance

If you did not do this problem last spring:
12.27: The onset of chaos: Full renormalization-group calculation

For everyone:

12.21: Diffusion equation and universal scaling functions.

12.21: Diffusion equation and universal scaling functions.

12.25: Avalanche size distribution.

?? Eigenvectors near the renormalization-group fixed point. ③

The critical exponents in the renormalization group are given by the eigenvalues of the RG transformation linearized near the fixed point. What do the eigenvectors mean?

Consider a two-dimensional Ising model with two parameters, a nearest-neighbor bond[1]

[1]Instead of thinking of a Hamiltonian space with temperature as an extra parameter, it is convenient to work at fixed temperature, and mimic raising temperature by lowering the overall scale of the energy.

$K = J/T$ and a next-neighbor interaction $K_2 = J_2/T$ lying along the diagonal bonds.

$$\begin{aligned}\mathcal{H} = &-K \sum_{i,j} S_{i,j}S_{i+1,j} + S_{i,j}S_{i,j+1} \\ &-K_2 \sum_{i,j} S_{i,j}S_{i+1,j+1} + S_{i,j}S_{i+1,j-1}\end{aligned} \tag{1}$$

If we decimate to the 'black' squares of a checkerboard (say, $i + j$ even), we get a new square-lattice Hamiltonian rotated by 45° coarse-grained by a factor $b = \sqrt{2}$. The next-neighbor bond basically becomes a nearest-neighbor bond – it mostly renormalizes to zero in one step, and contributes its value to the new nearest-neighbor coupling. The deviation of the nearest-neighbor bond from the critical point K^*, we may crudely assume, rescales by a factor $b^{1/\nu}$ under coarse-graining (remember $K \sim J/T$) and then is increased by K_2. So under one coarse-graining step

$$\begin{aligned}K' - K^* &= b^{1/\nu}(K - K^*) + K_2, \\ K_2' &= 0.\end{aligned} \tag{2}$$

(a) *Our crude renormalization-group flow is already linear. What is the fixed point? What is the Jacobian J about the fixed point? What are the eigenvalues λ_0 and λ_1?* (Let λ_1 be the relevant eigenvalue, greater than one.)

Our Jacobian matrix is not symmetric (or Hermitian), so it has two sets of eigenvectors – left eigenvectors $\hat{\boldsymbol{\ell}}_\alpha J = \lambda_\alpha \hat{\boldsymbol{\ell}}_\alpha$, and right eigenvectors $J\hat{\mathbf{r}}_\alpha = \lambda_\alpha \hat{\mathbf{r}}_\alpha$.

(b) *What are the left and right eigenvectors? Are the left eigenvectors orthonormal? Are they normal to the right eigenvector that has a different eigenvalue?*

(c) *Draw the flow in the (K, K_2) plane near the fixed point. Indicate the directions of the left eigenvectors and right eigenvectors in different colors. Also draw the boundary between the ferromagnetic and paramagnetic phase. How is this boundary related to the stable manifold of the fixed point? Is it related to any of the eigenvectors?*

Consider a new set of *scaling variables* u_α, given by the dot products of the displacement from the fixed point with the left eigenvectors:

$$u_\alpha = \boldsymbol{\ell}_\alpha \cdot (K - K^*, K_2) \tag{3}$$

(d) *Show that the phase boundary has $u_1 = 0$ (using the convention that λ_1 is the relevant direction). How do the coordinates u_α flow under the renormalization group?*

In general, there is a nonlinear transformation between the parameters $T, H, J_2, \ldots$ in a Hamiltonian and the natural coordinates $t(T, H, J_2), h(T, H, J_2), u(T, H, J_2)$ which flow simply under the renormalization group. This coordinate change is one of the contributors to analytic corrections to scaling.

(e) *Are u_0 and u_1 relevant, irrelevant, or marginal? Which coordinate, u_0 or u_1, is the scaling variable corresponding to the reduced temperature $t(K, K_2)$? If we write a*

property of our system $X(K, K_2) = X(K(u_0, u_1), K_2(u_0, u_1)) = u_1^x \mathcal{X}(u_0/u_1^y)$*, can there be any dependence on* u_0*, within our crude model? How does* X *vary near the phase boundary?*

References

[1] Sethna, J. P. and Myers, C. R. (2004). *Entropy, Order Parameters, and Complexity* computer exercises: Hints and software. http://www.physics.cornell.edu/sethna/StatMech/ComputerExercises.html.