Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
Material for Penultimate Week
Exercises due Thursday November 29
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Exercises

?? Supersymmetric harmonic oscillator.[1] (Quantum)③

One of the main predictions of supersymmetry[2] is that each particle comes with a supersymmetric partner with the same mass but with opposite statistics.[3] For example, the fermionic electron is paired with the bosonic selectron. Supersymmetry is also a potential symmetry of nature, with an unusual connection to the translational symmetries in space and time (the Poincaré group). Finally, supersymmetry allows one to calculate remarkable things about certain Hamiltonians. In this exercise, we shall explore a "zero-dimensional"[4] example of a supersymmetric Hamiltonian, and try to illustrate each of these features of supersymmetry.[5]

Remember the commutation relations for creation and annihilation operators suitable for bosons

$$[a, a^\dagger] = 1 \qquad [a, a] = [a^\dagger, a^\dagger] = 0, \tag{1}$$

and fermions

$$\{b, b^\dagger\} = 1 \qquad \{b, b\} = \{b^\dagger, b^\dagger\} = 0. \tag{2}$$

where $[A, B] = AB - BA$ is the commutator and $\{A, B\} = AB + BA$ is the anticommutator.[6] For this simple example, we take our bosons and fermions to be noninteracting,

[1] Developed in collaboration with John Stout, Fall 2013.

[2] The footnotes in this problem are meant as inspiration – tying it to fundamental ideas in theoretical physics. *None of the footnotes are necessary or useful for solving the problem* – ignore them if you wish.

[3] Supersymmetric partners have the same mass as long as supersymmetry is unbroken. We expect supersymmetry to be *spontaneously broken* at low energy scales, given that we have not yet detected any supersymmetric partners of the Standard Model particles.

[4] We often talk about quantum field theories in d spatial dimensions and one time dimension as d+1-dimensional field theories: our space-time is thus 3+1 dimensional. We can view non-relativistic quantum mechanics as a $d = 0$ quantum field theory, and it is in this regard that we consider the supersymmetric Hamiltonians described here as "zero-dimensional" or 0+1-dimensional.

[5] There are a number of discussions of the supersymmetric harmonic oscillator and zero-dimensional supersymmetry in the literature and on the Web. Feel free to consult these. If you find one particularly useful, reference it properly in your writeup.

[6] Be sure to avoid getting confused by our multiple uses of the terms 'boson' and 'fermion' in this exercise. There are really three different ways we use the terms, each extremely useful and compelling. They are:

(a) the objects which vibrate or have spins, that produce harmonic oscillators or two-state systems, so their creation and annihilation operators commute,

$$[a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0. \tag{3}$$

In one dimension, the Hamiltonian of the simple harmonic oscillator of frequency ω can be written either in terms of x and p:

$$\mathcal{H}_B = p^2/2m + \tfrac{1}{2}m\omega^2 x^2 \tag{4}$$

or in terms of the creation and annihilation operators

$$\mathcal{H}_B = \hbar\omega(a^\dagger a + \tfrac{1}{2}). \tag{5}$$

Here $\frac{1}{2}\hbar\omega$ is the ground state energy of the harmonic oscillator – the zero-dimensional analogue of the 'vacuum energy' in field theory.

The harmonic oscillator Hamiltonian can be written in a more symmetric way by using the anticommutator.

(a) *Show that $\mathcal{H}_B = \frac{1}{2}\hbar\omega\{a^\dagger, a\}$. Is the vacuum energy still $\frac{1}{2}\hbar\omega$?*

Note that we're now calling the ladder operators a and $a^\dagger$ 'creation' and 'annihilation' operators. In this new language, the n^{th} excited state of the harmonic oscillator can be viewed as a state with n bosons.

Define a 'fermionic harmonic oscillator' in analogy to the bosonic one, $\mathcal{H}_F = \frac{1}{2}\hbar\omega[b^\dagger, b]$. Again, we can view the n^{th} excited state as a state of n fermions.

(b) *What is the ground state energy of $\mathcal{H}_F$? How many fermions are in the ground state, in this new language? What is the energy of the state with one fermion?*

(c) *If we write the zero-fermion state as*[7] $|0\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ *and the one-fermion state as* $|1\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$*, then write b, $b^\dagger$, and $\mathcal{H}_F$ in terms of the three Pauli matrices σ_x, σ_y, and σ_z. Check that your form for b and $b^\dagger$ satisfy the anticommutation relations of eqn 2* (Remember $\sigma_x = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, $\sigma_y = \left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)$, and $\sigma_z = \left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$.)

We can write our first supersymmetric Hamiltonian by adding the boson and fermion harmonic oscillators:

$$\mathcal{H}_S = \mathcal{H}_B + \mathcal{H}_F = \tfrac{1}{2}\hbar\omega\left(\{a^\dagger, a\} + [b^\dagger, b]\right). \tag{6}$$

Note that the ground state energy for this Hamiltonian is zero.[8]

(b) the 'primitive' bosons (and fermions) which are excitations within a harmonic oscillator (e.g., N bosons = Nth excited state inside the vibrating object)

(c) the composite objects inside the supersymmetric Hamiltonian that merge zero or more 'primitive' bosons and fermions.

[7] Here we use the convention $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, instead of the notation used in quantum computing.

[8] That is a hint for part (b).

This supersymmetric Hamiltonian is not particularly difficult to solve. Because there is no interaction between the bosonic and fermionic parts of the Hamiltonian, the solution separates and the eigenstates are just products $\psi(x)\chi(s)$, and the energy of the eigenstate is the sum of the Fermi and Bose energies.

Remember that a composite particle with an odd number of primitive fermions is a fermion – so half of our eigenstates represent composite bosons, and half represent composite fermions.

(d) *Solve for the energies for the eigenstates of $\mathcal{H}_S$. Which eigenstates represent composite fermions? Which composite bosons? Draw the 'level diagram' for $\mathcal{H}_S$, with the first few composite boson eigenenergies as a column of horizontal lines on the left, and the first few composite fermion eigenenergies on the right. On each line, write the number of primitive bosons and fermions making up the composite. Is there a composite fermion state for each composite boson state? What state is the exception?* We shall hitherto drop the 'composite' label. If we interpret the energy of a state as the mass of a particle[9] , supersymmetry gives us for every fermion a boson with the same mass.

The fact that our Hamiltonian has (almost) one fermion state for each boson state is a result of an unusual symmetry of the Hamiltonian. To see this, let's define an operator, called the *supercharge*,

$$Q = b\left(\frac{p}{\sqrt{m}} + \mathrm{i}\sqrt{m}\omega x\right) = \mathrm{i}\sqrt{2\hbar\omega}ba^{\dagger}. \tag{7}$$

(Remember that $x = \sqrt{\hbar/2m\omega}(a^{\dagger} + a)$ and $p = \mathrm{i}\sqrt{m\omega\hbar/2}(a^{\dagger} - a)$.)

(e) *Show that* $[\mathcal{H}_S, Q] = 0$. (Hence Q is a symmetry of the Hamiltonian.) *Show that Q acting on a fermion state gives a constant times a boson state of the same energy, and that $Q^{\dagger}$ acting on a boson state almost always gives a constant times a fermion state of the same energy. Which of the ground states is the exception to this rule? Show that this ground state is an eigenfunction of Q and $Q^{\dagger}$ with eigenvalue zero.*[10]

Supersymmetry has been shown (by Haag, Lopuszanski, and Sohnius[11]) to be the only way to consistently extend the symmetries of spacetime. Spacetime has a spatial

[9] We can motivate this by remembering that we are dealing with a theory with zero spatial dimensions, and so the usual relativistic energy of a particle (which should correspond to a eigenstate of our Hamiltonian) $E = \sqrt{\mathbf{p}^2c^2 + m^2c^4}$ reduces to $E = mc^2$. We often interpret the mass of a particle as being the energy required to create a "copy" of the particle at rest, and it is analogous to the band gap energy in semiconductors.

[10] $Q\Psi = 0$ gives us a first-order differential equation which can be directly integrated to obtain this ground state wave function! This trick extends to field theory applications too – yet another way in which supersymmetry simplifies theorists lives.

[11] The story starts with the Coleman-Mandula no-go theorem in 1967. (According to n-Lab, a no-go theorem is "any theorem...that shows that an idea is not possible even though it may appear as if it should be." Thus Bell's theorem is a no-go theorem dictating the impossibility of local, hidden variable theories that reproduce the predictions of quantum mechanics.) The Coleman-Mandula theorem tells us that in a realistic quantum field theory, space-time symmetries (like the Lorentz group) can only be combined with internal symmetries (like the SU(3) of the strong interaction) in a trivial way (so that the total symmetry group is (space-time symmetry)×(internal symmetry group).

translational symmetry (with an associated conserved momentum), a time-translational symmetry (associated with the conserved energy, with the Hamiltonian giving the infinitesimal time-translation operator), and other symmetries (rotations and relativistic boosts). Combining these symmetries gives us the Poincaré group.

In our "zero-dimensional" harmonic oscillator, only the time-translational symmetry remains from the Poincaré group. How does supersymmetry extend time-translation invariance? Can we somehow create a time translation by supersymmetrically transforming it?

(f) *Show that* $\mathcal{H}_S = \frac{1}{2}\{Q, Q^\dagger\}$. We see that a combination of two supercharges generates a time translation!

The supersymmetric harmonic oscillator we looked at above may seem pretty trivial: how hard is it to get degenerate states when all states have the same energy splitting?

However, we can generate lots of interacting supersymmetric Hamiltonians by specifying a supercharge

$$Q_W = b\left(\frac{p}{\sqrt{m}} + \mathrm{i}\sqrt{m}W'(x)\right) \tag{8}$$

where $W'(x) = \mathrm{d}W/\mathrm{d}x$, and requiring that $\mathcal{H}_W = \frac{1}{2}\{Q_W, Q_W^\dagger\}$, where the real function $W(x)$ is called the *superpotential*.

Our Hamiltonian $\mathcal{H}_S$ can be viewed as the special case of $W(x) = \frac{1}{2}\omega x^2$. Note that our superpotential need not have units of energy.

(g) *Show that* Q_W *and* $\mathcal{H}_W$ *as 2×2 matrices*

$$\mathcal{H}_W = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix} \qquad \textit{and} \qquad Q_W = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}. \tag{9}$$

where the elements of the matrices are functions of p *and* x. (Hint: Remember $p = -\mathrm{i}\hbar\partial/\partial x$. You might check this against the Web, which has different units.)

There is a lovely relationship between the eigenvalues and eigenfunctions of $\mathcal{H}_1$ and $\mathcal{H}_2$, two seemingly different Hamiltonians. Let $\Psi_n^{(1)}(x)$ and $\Psi_m^{(2)}(x)$ be the n-th and m-th eigenfunctions of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

How did the no-go theorem go? You may remember, according to Noether's theorem, that all continuous symmetries are associated with conserved quantities: thus momentum and energy are the conserved quantities related to translations in space and time, and conversely $\mathbf{p}$ and $\mathcal{H}$ (or P^j and P^0 in four-vector notation) generate infinitesimal space and time translations. Coleman and Mandula showed that spacetime symmetry generators had to commute with generators of any new internal symmetries represented by commutation relations.

Haag, Lopuszanski, and Sohnius were able to skirt the Coleman-Mandula theorem by avoiding the hidden assumption that the new symmetry had to obey commutation relations: the new supersymmetries involve *anticommutation* relations. In fact, they were able to show that this is the *only* way of extending the Poincaré group for consistent, interacting quantum field theories with massive particles.

(h) *Using the fact that* $[\mathcal{H}_W, Q_W] = [\mathcal{H}_W, Q_W^\dagger] = 0$, *show that* $A^\dagger \Psi_m^{(2)}(x)$ *is an eigenstate of* $\mathcal{H}_1$ *and* $A\Psi_n^{(1)}(x)$ *is an eigenstate of* $\mathcal{H}_2$. (Thus, if we know the eigenfunctions and eigenenergies of one of the Hamiltonians, we know them for the other.)

Let us work out a specific example. Consider $W'(x) = (\pi\hbar/mL)\cot(\pi x/L)$.

(i) *Show that* $\mathcal{H}_1$ *is the particle-in-a-box Hamiltonian (Fig. 1) shifted by a constant to set its ground state energy to zero. Show that* $\mathcal{H}_2$ *is a Hamiltonian with potential*[12]

$$V(x) = \frac{\pi^2\hbar^2}{2mL^2}\left(2\csc^2\left(\frac{\pi x}{L}\right) - 1\right). \tag{10}$$

Using the first excited state $\Psi_2^{(1)}(x) = \sqrt{2/L}\sin(2\pi x/L)$ *of* $\mathcal{H}_1$ *and the operator* A, *generate the ground state of* $\mathcal{H}_2$ *and show that it is proportional to* $\sin^2(\pi x/L)$. *Explicitly show (taking the derivatives) that* $A\Psi_2^{(1)}(x)$ *is an eigenfunction of* $\mathcal{H}_2$ *and thus verify that its energy is the same as that of* $\Psi_2^{(1)}$.

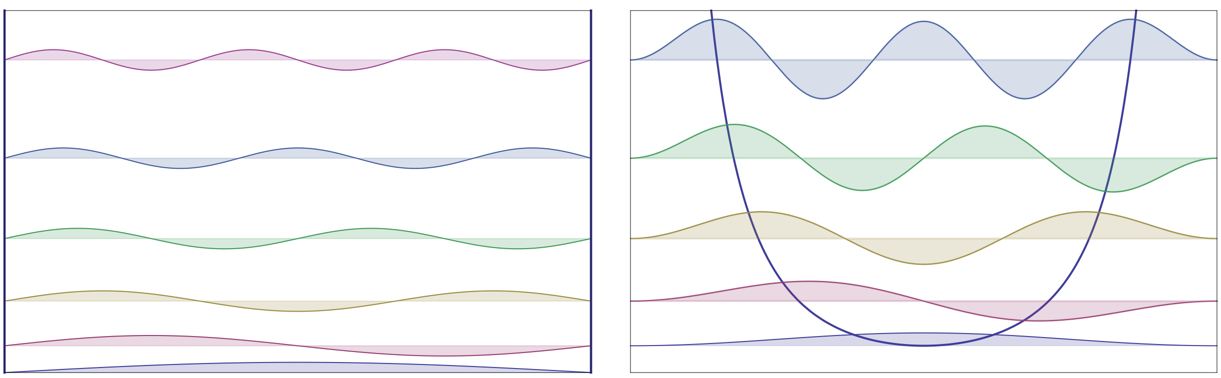

Fig. 1 Supersymmetric eigenenergies and eigenstates. (Left) Eigenstates for $\mathcal{H}_1$, the square well potential, displaced vertically by their eigenenergies. (Right) Eigenstates for $\mathcal{H}_2$, the $\csc^2 x$ potential, which is the supersymmetric pair for the square well.

While supersymmetry may not exist in nature, it has proved to be an excellent tool for gaining insight into the way theories with gauge symmetry behave. (For example, we have no proof that the strong interaction confines quarks, but Seiberg and Witten were able to demonstrate confinement in certain supersymmetric theories.) It also has allowed physicists to prove theorems in pure mathematics. Ed Witten, high-energy theorist at the Institute for Advanced Study, was awarded the Fields Medal (the Nobel equivalent in math) for his use of supersymmetry to figure out topological properties of a manifold (such as the Euler characteristic, related to the number of holes or handles a manifold has) by using the difference in the number of zero-energy 'fermion' and 'boson' wavefunctions on it.[13]

[12] Note that the $\Psi = 0$ boundary conditions for the two Hamiltonians are the same for both $\mathcal{H}_1$ and $\mathcal{H}_2$.

[13] E. Witten, *Supersymmetry and Morse Theory*, J. Diff Geom. **17**, 661692 (1982).

?? Quantum dissipation from phonons. (Quantum)②

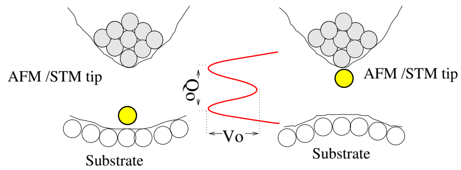

Fig. 2 Atomic tunneling from a tip. Any *internal* transition among the atoms in an insulator can only exert a force impulse (if it emits momentum, say into an emitted photon), or a force dipole (if the atomic configuration rearranges); these lead to non-zero phonon overlap integrals only partially suppressing the transition. But a quantum transition that changes the net force between two macroscopic objects (here a surface and a STM tip) can lead to a change in the net force (a force monopole). We ignore here the surface, modeling the force as exerted directly into the center of an insulating elastic medium.[14]See "Atomic Tunneling from a STM/AFM Tip: Dissipative Quantum Effects from Phonons" Ard A. Louis and James P. Sethna, *Phys. Rev. Lett.* **74**, 1363 (1995), and "Dissipative tunneling and orthogonality catastrophe in molecular transistors", S. Braig and K. Flensberg, *Phys. Rev. B* **70**, 085317 (2004).

Electrons cause overlap catastrophes (X-ray edge effects, the Kondo problem, macroscopic quantum tunneling); a quantum transition of a subsystem coupled to an electron bath ordinarily must emit an infinite number of electron-hole excitations because the bath states before and after the transition have zero overlap. This is often called an *infrared* catastrophe (because it is low-energy electrons and holes that cause the zero overlap), or an *orthogonality* catastrophe (even though the two bath states aren't just orthogonal, they are in different Hilbert spaces). Phonons typically do not produce overlap catastrophes (Debye–Waller, Frank–Condon, Mössbauer). This difference is usually attributed to the fact that there are many more low-energy electron-hole pairs (a constant density of states) than there are low-energy phonons ($\omega_k \sim ck$, where c is the speed of sound and the wave-vector density goes as $(V/2\pi)^3 \mathrm{d}^3 k$).

However, the coupling strength to the low energy phonons has to be considered as well. Consider a small system undergoing a quantum transition which exerts a net force at $x = 0$ onto an insulating crystal:

$$\mathcal{H} = \sum_k p_k^2/2m + 1/2\, m\omega_k^2 q_k^2 + F \cdot u_0. \tag{11}$$

Let us imagine a kind of scalar elasticity, to avoid dealing with the three phonon branches (two transverse and one longitudinal); we thus naively write the displacement of the atom at lattice site x_n as $u_n = (1/\sqrt{N}) \sum_k q_k \exp(-\mathrm{i}kx_n)$ (with N the number of atoms), so $q_k = (1/\sqrt{N}) \sum_n u_n \exp(\mathrm{i}kx_n)$.

Substituting for u_0 in the Hamiltonian and completing the square, find the displacement Δ_k of each harmonic oscillator. (Physically, the force F adds a small linear term *to the phonon mode with wavevector k, whose minimum becomes displaced by some amount Δ_k.) Let $|F\rangle$ be the ground state of the harmonic oscillators under the force F. Write the formula for the likelihood $\langle F|0\rangle$ that the phonons will all end in their ground states, as a product over k of the phonon overlap integral* $\exp(-\Delta_k^2/8\sigma_k^2)$ *(with $\sigma_k = \sqrt{\hbar/2m\omega_k}$ the zero-point motion in that mode). Converting the product to the exponential of a sum, and the sum to an integral $\sum_k \sim (V/(2\pi)^3 \int \mathrm{d}^3\mathbf{k}$, do we observe an overlap catastrophe?*

Note that you've calculated the probability of a zero-phonon transition – the likelihood that the quantum transition can happen without emitting any phonons is zero. But the same argument shows that there is zero probability of emitting one phonon, or any finite number of phonons. The only allowed transitions emit an infinite number of low-energy phonons. The initial and final ground states are in 'different Hilbert spaces' – no finite number of excitations can connect them.