Conformal Field Theory and $c-$Theorem

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These lecture notes give an elementary, intuitive, and hopefully self-contained (but by no means comprehensive) introduction to the basics of conformal field theory (CFT) in two dimensions. The aim is to develop just enough CFT from the ground up in order to understand and prove the $c-$theorem, which reveals the deep and subtle connection between entropy, CFT (fixed point in RG) and the direction of RG flow between them. The interested reader is referred to any of the excellent monographs listed in the reference. CFT is a minimalist art. Enjoy the ride!

Contents

1 The Scaling Limit: From Lattice To Field

We start with the regular Ising model on a $d-$dimensional lattice $\mathcal{D}$. Each lattice point has a spin $s(r) = \pm 1$ and any particular spin configuration appears with probability proportional to

$$\mathcal{W}(\{s(r)\}) = \exp(\sum_{r,r' \in \mathcal{D}} J(r - r')s(r)s(r')) \tag{1}$$

where the spin–spin interaction $J(r - r')$ is assumed to be short-range measured in lattice constant a. Same old story. Now we would like to do measurements on the lattice. We assume that any observable quantity on the lattice is local, which means that any measurement done at site $\vec{r}$, call it $\phi^{lat}(\vec{r})$, can only gather information of spins a few lattice sites away.

The quantity of interest to us is the lattice correlation function

$$\langle \phi_1^{lat}(\vec{r}_1)...\phi_n^{lat}(\vec{r}_n)\rangle_{\mathcal{D}} = \frac{1}{Z}\sum_{\{s(r)=\pm 1\}} \phi_1^{lat}(\vec{r}_1)...\phi_n^{lat}(\vec{r}_n)\mathcal{W}(\{s(r)\}) \tag{2}$$

where $Z = \sum_{\{s(r)=\pm 1\}} \mathcal{W}(\{s(r)\})$ is the partition function. Since all interactions are short-range and all observables are local, intuitively distant spins should not know each other's existence! The length scale where two distant spins de-correlate is given by the correlation length ξ, which can be formally defined by the exponential decay rate of the lattice correlation function above.

Now comes the big moment. What if we add more and more spins into the system by taking the lattice spacing $a \to 0$ while keeping the size of the domain $\mathcal{D}$ and correlation length ξ both fixed? Fixing $\mathcal{D}$ and ξ is absolutely crucial. In some sense we are doing the exact opposite of RG coarse-graining – introducing more microscopic degrees of freedom without changing the macroscopic correlations. Taking the *scaling limit* of the lattice model gives us a continuum field theory.

What can we possibly get by taking this limit? Scale invariance! To take the $a \to 0$ limit is to modulo out the shortest length scale so that the system has a chance to look self-similar when we zoom in arbitrarily closely. A familiar example is the Brownian motion, which is just the scaling limit of discrete random walk when both time and space intervals are scaled to zero.

2 Correlation Functions, Scaling Fields and OPE

What happens to the lattice correlation function under scaling limit? Most of them diverge. Why? Because the continuum limit of a lattice model is not itself a lattice model! Okay, fair enough. But what's the big deal? Well, in the lattice model we can define a perfectly fine observable $\phi_+^{lat}(\vec{r})$ which is the sum of all spins with distance ξ from the site $s(\vec{r})$. However, in the scaling limit this quantity is not well defined since it becomes the sum of infinitely many spins we stick into the system within ξ from $s(\vec{r})$! On the other hand, the *average* of all spins (i.e. magnetization) with distance ξ from $s(\vec{r})$ *is* well defined. The price we pay is that the fundamental degree of freedom is now a continuous function $\phi(\vec{r})$ instead of discrete spins $s(\vec{r})$.

This intuitive argument can be generalized. Though most lattice correlation functions diverge, there exist certain *linear combinations* of local lattice observables which are *multiplicatively renormalizable*, that is, they have a well-defined the continuum limit given by

$$\langle \phi_1(\vec{r}_1)...\phi_n(\vec{r}_n)\rangle_{\mathcal{D}} = \lim_{a\to 0} a^{-\sum_{i=1}^n x_i}\langle \phi_1^{lat}(\vec{r}_1)...\phi_n^{lat}(\vec{r}_n)\rangle_{\mathcal{D}} \tag{3}$$

where the quantities $\phi_j(\vec{r}_j)$ on LHS are called *scaling fields* (or scaling *operators*, especially in quantized versions of CFT) and the constants x_i is the *scaling dimension* of the operator ϕ_i. In case you haven't already noticed, these are exactly the quantities that appear in the plain old RG flow equation! The correlation function of scaling fields defined above will be the central quantity of interest in the subsequent discussion of CFT.

Exercise 2.1 We now perform our first conformal transform $r \to r' = br$ where the constant $b > 0$ is the familiar scale factor in RG. This also maps the domain $\mathcal{D} \to b\mathcal{D}$.

(a) Using only the definition of continuum correlation function (3), derive the power law scaling relation between the correlation functions $\langle \phi_1(b\vec{r}_1)...\phi_n(b\vec{r}_n)\rangle_{b\mathcal{D}}$ and $\langle \phi_1(\vec{r}_1)...\phi_n(\vec{r}_n)\rangle_{\mathcal{D}}$. Does this look familiar to an RG flow equation?

(b) Now consider a general mapping $\phi : \mathcal{D} \to \mathcal{D}'$ such that $r \to r' = \phi(r)$ is locally equivalent to composition of dilatation and rotation. Denote the Jacobian $b(r) = |\partial r'/\partial r| = |\phi'(r)|$. Guess the *conformal covariant* scaling relation between $\langle \phi_1(r'_1)...\phi_n(r'_n)\rangle_{\mathcal{D}'}$ and $\langle \phi_1(r_1)...\phi_n(r_n)\rangle_{\mathcal{D}}$. Your expression should agree with your result from (a) when b is a constant (i.e. uniform dilatation).

As with correlation functions in QFT, the correlation function $\langle \phi_1(\vec{r}_1)...\phi_n(\vec{r}_n)\rangle_{\mathcal{D}}$ is singular whenever $|\vec{r}_i - \vec{r}_j| \to 0$, keeping all other positions $\vec{r}_k$ fixed. You might think that the functional form of the correlation function has a rather subtle analytical structure – and you are exactly right! As well will see, all the fascinating properties of CFT are deeply rooted in the analytical structures of its correlation functions. In other words, the correlation functions contain all the necessary data to specify a CFT. Note the conspicuous absence of any particular Hamiltonian. How is this possible? Because each CFT actually represents a *universality class* of Hamiltonians! *Voila!*

So how do we see the analytical structures of correlation functions? The powerful tool we need is Operator Product Expansion (OPE), which states that

$$\langle \phi_i(\vec{r}_i)\phi_j(\vec{r}_j)...\rangle_{\mathcal{D}} = \sum_k C_{ij}^k(|\vec{r}_i - \vec{r}_j|)\ \langle \phi_k(\vec{r}_j)...\rangle_{\mathcal{D}} \tag{4}$$

where ... in the correlation function represents any arbitrary insertion of operators at positions sufficiently far from $\vec{r}_i$ and $\vec{r}_j$. In fact, it can be proved that OPE as defined above has its radius of convergence exactly at the next nearest operator insertion from $\vec{r}_i$ and $\vec{r}_j$, though this is not too important for our purpose since we are mainly interested in the singularity as $|\vec{r}_i - \vec{r}_j| \to 0$.

OPE is so commonly used in CFT that it has been chiseled into our reflex system. Note the striking similarity between OPE and Taylor expansion shown below (where we have dropped the other operator insertions in OPE for clarity). Both approximate a function evaluated at a nearby point $\vec{r}_i$ by an infinite sum of functions evaluated at the current point $\vec{r}_j$. Their difference is also significant. The OPE coefficients $C_{ij}^k(|\vec{r}_i - \vec{r}_j|)$ depend on the types of both operators ϕ_i and ϕ_j in the product as well as the new operator ϕ_k it brings to the infinite sum.

$$\phi_i(\vec{r}_i)\phi_j(\vec{r}_j) = \sum_k C_{ij}^k(|\vec{r}_i - \vec{r}_j|)\ \phi_k(\vec{r}_j) \tag{5}$$

$$f(x_i) = \sum_{n=0}^{\infty} \frac{(x_i - x_j)^n}{n!} f^{(n)}(x_j) \tag{6}$$

It is interesting to note that OPE closely resembles the Lie algebra of nonabelian Lie group $\phi_i\phi_j = \sum_k C_{ij}^k\ \phi_k$ with C_{ij}^k being the familiar structure constant. Also, the choice we made to evaluate ϕ_k at $\vec{r}_j$ on RHS of OPE is completely arbitrary. An equally valid choice is the midpoint $(\vec{r}_i + \vec{r}_j)/2$ or essentially anywhere near both $\vec{r}_i$ and $\vec{r}_j$ but far from the other operator insertions.

Exercise 2.2 Consider the same conformal transform $r \to r' = br$. Using OPE (4) and the scaling relation of correlation functions you derived in the previous exercise, derive the scaling relation of OPE coefficients $C_{ij}^k(b|\vec{r}_i - \vec{r}_j|)$ and $C_{ij}^k(|\vec{r}_i - \vec{r}_j|)$. Deduce that the OPE expansion looks like

$$C_{ij}^k(|\vec{r}_i - \vec{r}_j|) = \frac{c_{ij}^k}{|\vec{r}_i - \vec{r}_j|^{\nu_{ijk}}} \tag{7}$$

Find the expression of the critical exponent ν_{ijk} with the scaling dimensions x_i, x_j, and x_k.

You have just shown that given a CFT, the two-point function is completely fixed by the *universal* OPE coefficients c_{ij}^k and scaling dimensions! Nowhere have we used Lagrangians, Gaussian integrals, Feynman diagrams, or any small-parameter expansions, and yet the result (7) is exact and nonperturbative! I hope you are impressed so far. But the real magic is just about to begin.

3 CFT in 2 Dimensions

We now restrict our discussion of CFT in two dimensions. A conformal transformation f is a map between domains $f : \mathcal{D} \to \mathcal{D}$ such that the metric tensor is invariant up to an overall position-dependent factor $g_{\mu\nu}(r) \to \Omega(r) g_{\mu\nu}(r)$. An immediate consequence is that at any point $p \in \mathcal{D}$, the angle formed by tangent vectors of any two curves intersecting at p is preserved by the transform.

3.1 2D Conformal $\iff$ Holomorphic

It turns out that conformal transforms in $d = 2$ correspond to holomorphic functions on the complex plane $\mathbb{C}$. So see this, it's better to use the complex coordinates

$$z = r^1 + ir^2 \qquad \bar{z} = r^1 - ir^2 \tag{8}$$

and more generally any vector field $\vec{v}(\vec{r}) = (v^1, v^2)$ can be written as

$$v^z(z, \bar{z}) = v^1(\vec{r}) + iv^2(\vec{r}) \quad v^{\bar{z}}(z, \bar{z}) = v^1(\vec{r}) - iv^2(\vec{r}) \tag{9}$$

where z and $\bar{z}$ on v^z and $v^{\bar{z}}$ are called *conformal indices*. Why do we need a separate definition of $v^{\bar{z}}$ at all, since it is just the complex conjugate of v^z? In fact we don't. For most purposes $v^{\bar{z}}$ carries no additional physical information than v^z but rather serves as a convenient notation which tremendously simplifies the tensor expressions, as we shall see. The metric becomes

$$ds^2 = (dr^1)^2 + (dr^2)^2 = \left(\frac{dz + d\bar{z}}{2}\right)^2 + \left(\frac{dz - d\bar{z}}{2i}\right)^2 = dz d\bar{z} \tag{10}$$

so we can read off the metric tensor in conformal indices (μ, ν):

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \tag{11}$$

Note that we must be careful when raising and lowering tensor indices since it introduces factors of 2 and swaps the conformal indices:

$$v_z = g_{z\mu} v^\mu = \frac{1}{2} v^{\bar{z}} \qquad v_{\bar{z}} = g_{\bar{z}\mu} v^\mu = \frac{1}{2} v^z \tag{12}$$

Finally we define the derivative operators

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2) \equiv \partial \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2) \equiv \bar{\partial} \tag{13}$$

so that $\partial z = \bar{\partial}\bar{z} = 1$ and $\bar{\partial} z = \partial \bar{z} = 0$. We will use ∂ and $\bar{\partial}$ exclusively from now on.

Exercise 3.1.1 Let's now stretch our conformal muscles! Consider the general transform $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$ in $\mathbb{R}^2$ where $\alpha^\mu(r)$ is infinitesimal. It turns out that such mapping is conformal transform iff the local *shear component* vanishes, i.e.

$$\alpha^{\mu,\nu} + \alpha^{\nu,\mu} - \alpha^{\lambda,}{}_{\lambda}\, g^{\mu\nu} = 0 \tag{14}$$

where $\alpha^{\mu,}{}_{\nu} = \partial \alpha^\mu(r)/\partial r^\nu$. (Caution: watch out the indices!) Write (14) in conformal indices to get four equations. Show that two of them are trivially satisfied while the other two are equivalent to

$$\bar{\partial}\alpha^z = 0, \qquad \partial \alpha^{\bar{z}} = 0 \tag{15}$$

You have just shown that the infinitesimal mapping α is conformal iff α^z is holomorphic (i.e, $\bar{\partial}\alpha^z = 0$)! Very nice, isn't it? The generalization to finite case is straightforward: $z \to z' = f(z)$ is conformal iff f is holomorphic since $ds^2 = dz d\bar{z} \to df(z) d\bar{f}(\bar{z}) = \partial f(z) \bar{\partial}\bar{f}(\bar{z}) dz d\bar{z} = \Omega(z,\bar{z}) dz d\bar{z}$.

3.2 Complex Scaling Dimensions

Let's start from the simplest finite conformal transform in 2D: uniform dilatation and rotation. In complex coordinates z and $\bar{z}$, this cooresponds to multiplication by a constant complex number $z \to z' = be^{i\theta} z \equiv \lambda z$ where $b, \theta \in \mathbb{R}$ are constant. Consequently $\bar{z} \to be^{-i\theta}\bar{z} = \bar{\lambda}\bar{z}$.

Consider any operator $\phi_j(z, \bar{z})$. We say that ϕ_j has *conformal spin* s_j (which bears no relation to spins of elementary particles) if it transforms under $z \to \lambda z$ as

$$\phi_j(z,\bar{z}) \to e^{is_j\theta}\phi_j(\lambda z, \bar{\lambda}\bar{z}) = b^{-x_j}\phi_j(z,\bar{z}) \tag{16}$$

where x_j is the usual scaling dimension of ϕ_j defined in (3). Since a single $\lambda \in \mathbb{C}$ comes in more handy to describe this transform, we would like to define the above scaling relation as

$$\phi_j(\lambda z, \bar{\lambda}\bar{z}) = \lambda^{-\Delta_j}\bar{\lambda}^{-\bar{\Delta}_j}\phi_j(z,\bar{z}) \tag{17}$$

Comparing the two equations, we find that $x_j = \Delta_j + \bar{\Delta}_j$ and $s_j = \Delta_j - \bar{\Delta}_j$, where the doublet $(\Delta_j, \bar{\Delta}_j)$ are called the *complex scaling dimensions* of ϕ_j. This is another unfortunate nomenclature (together with *second* quantization and renormalization *group*) which serves to confuse more than elucidate its physical meaning since $(\Delta_j, \bar{\Delta}_j)$ are both *real* numbers!

In essence, we are just trading $(x_j, s_j) \to (\Delta_j, \bar{\Delta}_j)$. What did we gain from this trade? Well, doesn't (17) look awfully like the plain old scaling equation $\phi_j(br_j) = b^{-x_j}\phi_j(r_j)$? Yeah, as you probably have guessed by now, $(\Delta_j, \bar{\Delta}_j)$ allow us to generalize the scaling relation to arbitrary conformal transforms $z \to z' = f(z)$ in a very natural, elegant, and easy-to-remember way:

$$\langle \phi_1(z_1', \bar{z}_1')...\phi_n(z_n', \bar{z}_n')\rangle = \prod_{j=1}^{n} (\partial f(z_j))^{-\Delta_j} (\bar{\partial}\bar{f}(\bar{z}_j))^{-\bar{\Delta}_j} \langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle \tag{18}$$

3.3 Stress Tensor

So far our discussion have been based on abstract notions of operators ϕ_j which could be describing the exotic physics of strings, ether or evil little aliens. Now we introduce our first concrete operator – the stress tensor – which brings back the long-lost concepts of energy and momentum.

Consider again an infinitesimal transform $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$, where α is not necessarily conformal. The stress tensor is defined by the change of the action δS under this transform:

$$\delta S \equiv -\frac{1}{2\pi}\int T_{\mu\nu}\alpha^{\mu,}{}_{\nu} d^2 r = \frac{1}{2\pi}\int (\partial^\nu T_{\mu\nu})\alpha^\mu(r) d^2 r \tag{19}$$

where we integrate by parts in the second equality. If $\alpha^\mu(r)$ corresponds to a symmetry of the system, $\delta S = 0$. The transformations of particular interest to us are:

Translation: $\alpha^\mu(r) = a^\mu$ is constant, then $\delta S = 0 \implies \partial^\nu T_{\mu\nu} = 0$ (conserved).

Rotation: $\alpha^\mu(r) = \omega^\mu_\nu r^\nu$ where $\omega^\mu_\nu + \omega^\nu_\mu = 0$, then $\delta S = 0 \implies T_{\mu\nu} = T_{\nu\mu}$ (symmetric).

Dilataion: $\alpha^\mu(r) = b r^\mu = b g^{\mu\nu} r_\nu$, then $\delta S = 0 \implies T_{\mu\nu} g^{\mu\nu} = T^\mu_\mu = 0$ (traceless).

So what does the stress tensor look like if we use the conformal indices? Well, since $T_{\mu\nu}$ is a rank $(0,2)$ tensor, it transforms covariantly under the coordinate transform $(x, y) \to (z, \bar{z})$:

$$T(z, \bar{z}) \equiv T_{zz} = \frac{\partial x^\mu}{\partial z}\frac{\partial x^\nu}{\partial z} T_{\mu\nu} = \frac{1}{4}(T_{xx} - T_{yy}) + \frac{1}{4i}(T_{xy} + T_{yx}) \tag{20}$$

$$\bar{T}(z, \bar{z}) \equiv T_{\bar{z}\bar{z}} = \frac{\partial x^\mu}{\partial \bar{z}}\frac{\partial x^\nu}{\partial \bar{z}} T_{\mu\nu} = \frac{1}{4}(T_{xx} - T_{yy}) - \frac{1}{4i}(T_{xy} + T_{yx}) \tag{21}$$

$$T_{z\bar{z}} = \frac{1}{4}(T_{xx} + T_{yy}) + \frac{1}{4i}(T_{xy} - T_{yx}) \tag{22}$$

$$T_{\bar{z}z} = \frac{1}{4}(T_{xx} + T_{yy}) - \frac{1}{4i}(T_{xy} - T_{yx}) \tag{23}$$

What on earth just happened? The stress tensor in conformal indices becomes a crazy complex linear combination of its Eucliedean components and loses the usual "energy" or "momentum" interpretation. It might help to understand this as the *nirvana* of the stress tensor in CFT, where we sacrifice the physical "energy–momentum" in order to conjure up the bewildering conformal magic of $T(z)$, which is just about to take stage. Before we move on, take a moment to convince yourself that under rotation by $\pi/2$ (or equivalently $z \to iz$), we have $T_{zz} \to -T_{zz}$ and $T_{z\bar{z}} \to T_{z\bar{z}}$.

Exercise 3.3.1 Let's derive an important property of the stress tensor in two dimensions. Write down the above three constraints of the stress tensor with conformal indices T_{zz}, $T_{z\bar{z}}$, etc. Show that (1) $T_{\mu\nu}$ is symmetric and traceless in Euclidean coordinates $\implies T_{z\bar{z}} = T_{\bar{z}z} = 0$ and (2) $T_{\mu\nu}$ is conserved $\implies \bar{\partial}T_{zz} = \partial T_{\bar{z}\bar{z}} = 0$. This shows the stress tensor is diagonal, $T(z) \equiv T_{zz}$ is a holomorphic function and $\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$ is antiholomorphic.

Exercise 3.3.2 Let's do a quick sanity check with a free Gaussian scalar field in $\mathbb{R}^2$:

$$S[h] = \frac{1}{4\pi}\int d^2r\ \partial_\mu h\ \partial^\mu h \tag{24}$$

(a) Show that the action $S[h]$ is invariant under global translation, rotation, and dilatation (hence it's a Gaussian fixed point under RG flow). Then find the equation of motion of the field h.

(b) Find the stress tensor $T_{\mu\nu}$ in Euclidean coordinates by

$$T_{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu h)}\partial_\nu h - g_{\mu\nu}\mathcal{L} \tag{25}$$

(c) Compute $T(z)$ and $\bar{T}(\bar{z})$ and show that the equation of motion of h implies that $\bar{\partial}T(z) = \partial\bar{T}(\bar{z}) = 0$.

3.4 Conformal Ward Identity

Now we develop the final piece of the CFT machinery we need – the conformal Ward identity – which reveals the elegant analytic structure of the stress tensor. Recall that a holomorphic function can be expanded around a fixed point $z_0 \in \mathbb{C}$ in its *Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \tag{26}$$

It is helpful to think of the Laurent series as a *vector representation* of the function f with the basis vectors $(z - z_0)^n$ and coefficients a_n. By Cauchy Integral Theorem, we can project out any coefficient a_n via the contour integral along any closed contour γ which contains z_0

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)dz}{(z - z_0)^{n+1}} \tag{27}$$

Note that the holomorphic function (i.e. conformal transform) f has so much regularity in its structure that all of its behavior inside any region Ω is encoded by its values at the boundary $\partial\Omega$!

Now consider one such open and bounded region $\Omega \subset \mathcal{D}$ which contains all the points $z_1, ..., z_n$ of an $n-$point correlation function $\langle\phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle$. We would like to construct an infinitesimal transform $r^\mu \to r'^\mu = r^\mu + \alpha^\mu(r)$ which is conformal inside Ω and identity inside its complement $\Omega^c = \mathcal{D}\backslash\Omega$. Since $\alpha^{\mu,}{}_\nu = 0$ outside Ω, we have

$$\begin{aligned}\delta S &= -\frac{1}{2\pi}\int_{\mathcal{D}} T_{\mu\nu}\alpha^{\mu,}{}_\nu d^2r = -\frac{1}{2\pi}\int_\Omega T_{\mu\nu}\alpha^{\mu,}{}_\nu d^2r \\ &= -\frac{1}{2\pi}\int_\Omega \partial^\nu(T_{\mu\nu}\alpha^\mu)d^2r + \frac{1}{2\pi}\int_\Omega(\partial^\nu T_{\mu\nu})\alpha^\mu(r)d^2r\end{aligned} \tag{28}$$

Since α is conformal inside Ω, the stress tensor is conserved $\partial^\nu T_{\mu\nu} = 0$. Hence the second term vanishes. In the first term, we are integrating a total derivative over a finite region Ω with boundary. Using the divergence theorem, we have

$$\begin{aligned}\delta S &= -\frac{1}{2\pi}\int_\Omega \partial^\nu (T_{\mu\nu}\alpha^\mu) d^2 r \\ &= -\frac{1}{2\pi}\int_{\partial\Omega} T_{\mu\nu}\alpha^\mu n^\nu dl \\ &= -\frac{1}{2\pi i}\int_{\partial\Omega} T(z)\alpha(z)dz + c.c.\end{aligned} \tag{29}$$

where $n^\nu(r)$ is the outward pointing normal vector of the boundary $\partial\Omega$ and in the last equality we have written δS in complex coordinates with the usual definition $\alpha(z) \equiv \alpha^1(r) + i\alpha^2(r)$. Stare at the last equality in (29) and sense the sweet scent of Cauchy Integral Theorem in the air!

The infinitesimal transform $\alpha^\mu(r)$ also generates an infinitesimal change (i.e. linear response) in the correlation function which is formally defined via the path integral

$$\begin{aligned}\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle &= \frac{1}{Z}\int \mathcal{D}h\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\ e^{-S_0[h]-\delta S} \\ &\approx \frac{1}{Z}\int \mathcal{D}h\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\ e^{-S_0[h]}(1-\delta S) \\ &= \langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle_0 + \delta\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle\end{aligned} \tag{30}$$

hence the infinitesimal change to the correlation function is

$$\begin{aligned}\delta\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle &\approx \frac{1}{Z}\int \mathcal{D}h\ (-\delta S)\phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\ e^{-S_0[h]} \\ &= -\langle \delta S\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle_0 \\ &= \frac{1}{2\pi i}\int_{\partial\Omega} \alpha(z)\langle T(z)\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle dz + c.c.\end{aligned} \tag{31}$$

Exercise 3.4.1 We are almost ready to derive the conformal Ward identity! There are two extra pieces of ingredients we need:

(a) If we take $\alpha(z) = \epsilon \in \mathbb{C}$ to be a constant infinitesimal translation, show that

$$\frac{1}{2\pi i}\int_{\partial\Omega} \langle T(z)\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle dz = \sum_{j=1}^n \partial_{z_j}\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle \tag{32}$$

(b) If we take $\alpha(z) = \lambda(z - z_j)$ to be a composition of rotation and dilatation, show that

$$\frac{1}{2\pi i}\int_{\partial\Omega} (z-z_j)\langle T(z)\ \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle dz = \Delta_j\langle \phi_1(z_1,\bar{z}_1)...\phi_n(z_n,\bar{z}_n)\rangle \tag{33}$$

where $\lambda \in \mathbb{C}$ is infinitesimal and you might find the conformal scaling relation (18) very useful.

(c) Using the results from part (a) and (b) as well as Cauchy Integral Theorem, show that the OPE of $T(z)$ with any operator $\phi_j(z_j, \bar{z}_j)$, if treated as a single holomorphic function $f(z) \equiv T(z)\phi_j(z_j, \bar{z}_j)$, must contain the following singular terms in its Laurent series expansion

$$\langle T(z)\ \phi_j(z_j, \bar{z}_j)...\rangle = \Big(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{(z-z_j)}\partial_{z_j} + ...\Big)\langle \phi_j(z_j, \bar{z}_j)...\rangle \tag{34}$$

where ... in the sum represents other possible terms in the Laurent series of the form $(z-z_j)^n$.

Now if we assume that the *most* singular terms in the Laurent series expansion (34) is $\mathcal{O}((z-z_j)^{-2})$ for all operators $\phi_j(z_j, \bar{z}_j)$, then the correlation function $g(z) \equiv \langle T(z)\ \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle$ is a meromorphic function of z in Ω and is completely determined by its singularities

$$\langle T(z)\ \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle = \sum_{j=1}^{n}\Big(\frac{\Delta_j}{(z-z_j)^2} + \frac{1}{(z-z_j)}\partial_{z_j}\Big)\langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle \tag{35}$$

This is the conformal Ward identity! All operators such that the most singular term in their OPE with $T(z)$ is $\mathcal{O}((z-z_j)^{-2})$ are called *primaries* while those with less singular leading terms in their OPE with $T(z)$ are called *descendants*. We have derived the conformal Ward identity as a consequence of path integral under a given action S. What if we don't know (or even care) about S? Well, we just reverse the logic and use the conformal Ward identity (35) to *define* $T(z)$!

Exercise 3.4.2 The conformal Ward identity has an important implication:

(a) Inserting the Ward identiy (35) back into the integral on the RHS of (31) and evaluating the contour integral, show that for a general infinitesimal transform $\alpha(z)$, we have

$$\delta\langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle = \sum_{j=1}^{n}(\alpha'(z_j)\Delta_j + \alpha(z_j)\partial_{z_j})\langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle + c.c. \tag{36}$$

(b) Deduce that for a finite conformal transform $z \to z' = f(z)$, we have

$$\langle \phi_1(z_1, \bar{z}_1)...\phi_n(z_n, \bar{z}_n)\rangle_{\mathcal{D}} = \prod_{j=1}^{n}(\partial f(z_j))^{\Delta_j}(\bar{\partial}\bar{f}(\bar{z}_j))^{\bar{\Delta}_j}\langle \phi_1(z'_1, \bar{z}'_1)...\phi_n(z'_n, \bar{z}'_n)\rangle_{\mathcal{D}'} \tag{37}$$

which we recognize is nothing but the conformal scaling relation (18)! This is a crucial caveat which we have previously glossed over: the conformal scaling relation only holds for *primary* operators!

3.5 Central Charge

We have seen that the stress tensor $T(z)$ plays a major role in fixing the structure of CFT. The OPE $\langle T(z)\phi_j...\rangle$ induces the notion of primary and descendant operators and generates the scaling dimension Δ_j. But is $T(z)$ itself a primary operator? Answer: NO! In fact, the OPE is

$$T(z)T(z_0) = \frac{c/2}{(z-z_0)^4} + \frac{2}{(z-z_0)^2}T(z_0) + \frac{1}{(z-z_0)}\partial T(z_0) + ... \tag{38}$$

where the constant $c \in \mathbb{R}$ is the *conformal anomaly number* or the *central charge* of the CFT. Note that if $c = 0$, $T(z)$ would have been primary. However, for any general CFT, $c \neq 0$. It turns out that knowing the central charge c, together with the complex scaling dimensions $(\Delta_j, \bar{\Delta}_j)$ and the OPE coefficients c_{ij}^k, is sufficent to completely specify the analytic structure of CFT!

Before we move on, we can repeat the same analysis as in **Exercise 3.4.2** and deduce that for an infinitesimal transform $z \to z' = z + \alpha(z)$, the linear response $\delta T(z)$ is

$$\delta T(z) = 2\alpha'(z)T(z) + \alpha(z)\partial T(z) + \frac{c}{12}\alpha'''(z) \tag{39}$$

and for a finite conformal transform $z \to z' = f(z)$, the transform of $T(z) \to T(z')$ is given by

$$T(z) = (\partial f(z))^2 T(z') - \frac{c}{12}\{f(z), z\} \tag{40}$$

where the last term $\{f(z), z\}$ is the *Schwarzian derivative* of the function f (Ouch!):

$$\{f(z), z\} \equiv \frac{f'''(z)f'(z) - \frac{3}{2}f''(z)^2}{f'(z)^2} \tag{41}$$

Exercise 3.5.1 Who ordered the central charge c? In this exercise we develop some physical intuition of the central charge and compute the Schwarzian derivative of a simple conformal transform $f(z) = (2\pi/L)\log(z)$, where $\log(z) = \log|z| + iArg(z)$ and $0 \leq Arg(z) < 2\pi$. Note that f maps the entire complex plane onto a strip Σ with infinite length in the real direction u and finite width L in the imaginary direction v. Don't forget to impose the periodic boundary condition in the v direction!

(a) Calculate the Schwarzian derivative $\{f(z), z\}$. In this case it should be quite simple!

(b) Show that in the complex plane, $\langle T(z)\rangle_{\mathbb{C}} = \langle \bar{T}(\bar{z})\rangle_{\mathbb{C}} = 0$ by rotational invariance $z \to e^{i\theta}z$.

(c) Deduce from the transformation rule (40) that the corresponding quantity on the strip Σ is

$$\langle T(z)\rangle_\Sigma = \langle \bar{T}(\bar{z})\rangle_\Sigma = \frac{c}{24}\left(\frac{2\pi}{L}\right)^2 \tag{42}$$

(d) Show that the reduced free energy per unit length of the strip is

$$f_0(L) = -\frac{1}{2\pi}\int_0^L \langle T_{uu}\rangle_\Sigma \, dv = -\frac{\pi c}{6L} \tag{43}$$

(e) By imaginary-time path integral, we can treat the $L \times \infty$ strip as a one-dimensional quantum system at finite temperature $k_B T = 1/L$. Your answer from part (d) is then the reduced free energy per unit length $f/(k_B T)$ of this quantum system. Show that the specific heat is given by

$$C \sim \frac{\pi c k_B^2}{3} T \tag{44}$$

(f) Based on your results above, can you explain why the central charge c is effectively counting the gapless degrees of freedom of the CFT?

Here comes the real magic: if $c < 1$, the possible values of central charge c must have the form

$$c = 1 - \frac{6}{m(m+1)} \tag{45}$$

where $m \geq 3$ and $m \in \mathbb{Z}^+$ [5]. This (almost) magical constraint is a deep consequence of unitarity and classifies all possible CFT with $c < 1$. In addition, the corresponding complex scaling dimensions of the primary operators are completely classfied by the *Kac formula*:

$$\Delta = \Delta_{r,s} = \frac{(r(m+1) - sm)^2 - 1}{4m(m+1)} \tag{46}$$

where $1 \leq s \leq m$ and $1 \leq r \leq m-1$. The corresponding CFT is called a *unitary minimal model.*

For the critical 2D Ising model, $m = 3$. Hence $c = 1/2$. If we take $r = 2$ and $s = 1$, we get $\Delta_{2,1} = 1/2 = \Delta_{1,3}$. If we take $r = s = 2$, we get $\Delta_{2,2} = 1/16 = \Delta_{1,2}$. If we instead look up the scaling dimensions, we find that $\Delta_\sigma = 1/8 = 2\Delta_{2,2}$ and $\Delta_\epsilon = 1 = 2\Delta_{2,1}$. Why the extra factor of 2? Well, each $\Delta_{r,s}$ is counted twice due to the inherent symmetry $\Delta_{r,s} = \Delta_{m-r,m+1-s}$ in the Kac formula. The complex scaling dimension of each *physical* operator is the sum of two degenerate scaling dimensions classified by the Kac formula. The tricritical Ising model has $m = 4$ and the 3-state Potts model has $m = 5$. Moreover, we know that (45) gives a valid CFT for each $m \geq 3$ without even knowing any of its microscopic realizations! CFT indeed has *too much* structure!

4 The $c-$Theorem

4.1 Motivation

The basic CFT machinery we have developed so far already allows us to state and prove the fascinating $c-$theorem, which resolves the question: can RG flows form closed orbits (from fixed point A to another fixed point B and then *back* to A)? Intuitively this cannot happen since coarse–graining effectively throws away (or averages over) short–distance degrees of freedom. How can these UV degrees of freedom re–emerge in the long distance again? In fact, it is not even clear what we mean by *short* distance any more! Since to get from A to B, we can either keep coarse–graining or *anti*–coarse–grain backwards in time. If a weary traveler falls asleep at A and wakes up at B, he cannot tell whether he has gone forward to the past or backward to the future[1]!

So what goes wrong if we allow time to go both ways? Entropy! Since the total entropy in any isolated system can never decrease in "time", our traveler can always tell how he has got from A to B (either forward or backward in time) by measuring the entropy. Indeed, this intuitive argument has already touched on the very essence of the $c-$theorem, which (in its full glory) states that

Theorem: There exists a function $C(\{K\})$ of the coupling parameters $\{K\}$ such that it is
(1) monotonically decreasing along any RG flow trajectory;
(2) stationary only at the fixed points $\{K_c\}$;

(3) equal to the *central charge* of the corresponding CFT at each fixed point.

In **Exercise 3.5.1** we showed that the central charge c is effectively counting the degrees of freedom (hence the entropy) in the system. Therefore, what the $c-$theorem really says is that coarse–graining *always* lowers the entropy of the system. Hence RG flows must point from higher entropy to lower entropy fixed points. It clearly follows that there cannot be closed orbits. Duh!

4.2 Proof via Stress Tensor

Although physically intuitive, the rigorous proof of the $c-$theorem poses extremely difficult challenges in general dimensions. Miraculously, the rich analytic structure of correlation functions in two–dimensional CFT yields a simple, elegant, and almost minimalist proof, provided that the theory satisifes three additional constraints *everywhere* along the RG trajectory:

Translational Invariance $\implies \partial^\nu T_{\mu\nu} = 0 \leftrightarrow$ energy and momentum conservation

Rotational Invariance $\implies \langle \phi(z,\bar{z}) \rangle = 0$ and $T_{\mu\nu} = T_{\nu\mu}$

Reflection Positivity $\implies \langle \phi_i(z_1,\bar{z}_1)\ \phi_i(z_2,\bar{z}_2) \rangle \geq 0$

It is critical to note that in our formulation of the $c-$theorem, every point along the RG flow is a *continuum* field theory, where we have already taken the lattice constant $a \to 0$ in the correpond-ing lattice model. Also, invariance under translation and rotation further rules out intrinsically anisotropic systems (such as those with a spatially dependent magnetic field or potential). You might think that this is too restrictive, since in practice there are also irrelevant, anisotropic op-erators. However, remember that the point of the $c-$theorem is to establish whether an RG flow between two fixed points $A \to B$ is *possible* at all. If $A \to B$ is impossible through isotropic operators, it is certainly impossible if we turn on any irrelevant anisotropic operators!

Proof. The only ingredients we need are the stress tensor $T = T_{zz}$, its trace $\Theta(z,\bar{z}) = T^z_z + T^{\bar{z}}_{\bar{z}} = 4T_{z\bar{z}}$, as well as their two-point functions $\langle TT \rangle$, $\langle \Theta\Theta \rangle$ and $\langle \Theta T \rangle$. Note that $\Theta(z,\bar{z})$ is non-zero along the RG trajectory since the system is no longer scale invariant. We will still use OPE, though the coefficients $C^k_{ij}(|z_i - z_j|)$ are no longer simple power laws

$$C^k_{ij}(|z_i - z_j|) = \frac{c^k_{ij}}{|z_i - z_j|^{\nu_{ijk}}} \tag{47}$$

but rather general functions $F(|z_i - z_j|, \{K\})$ which depend on the couplings $\{K\}$ along RG flow (which we shall suppress for clarity). Hence the two-point function $\langle TT \rangle$ can be written as

[1] "So we beat on, boats against the current, borne back ceaselessly into the past." –*The Great Gatsby*

$$
\begin{aligned}
\langle T(z,\bar z)\ T(0,0)\rangle &= \Big\langle \frac{c/2}{z^4} + \frac{\Delta_T}{z^2}T(z_0) + \frac{1}{z}\partial T(z_0) + \dots \Big\rangle \\
&\to \Big\langle \frac{F(z\bar z)}{z^4} + \frac{F_1(z\bar z)}{z^2}T(z_0) + \frac{F_2(z\bar z)}{z}\partial T(z_0) + \dots \Big\rangle \\
&= \frac{F(z\bar z)}{z^4} + \frac{F_1(z\bar z)}{z^2}\langle T(z_0)\rangle + \frac{F_2(z\bar z)}{z}\langle \partial T(z_0)\rangle + \dots \\
&= \frac{F(z\bar z)}{z^4}
\end{aligned}
\tag{48}
$$

where we start from $\langle TT\rangle$ of a CFT (38) in the first equality and generalize it to any point along RG flow by replacing $c/2 \to F(z\bar z)$, $\Delta_T \to F_1(z\bar z)$, etc. Note that rotational symmetry enforces z^4 in the denominator (to make manifest the conformal spin $s_T = 2$) and kills all $\langle \partial^n T\rangle$ terms in the OPE. Similarly, since the trace $\Theta(z,\bar z)$ has conformal spins $s_\theta = 0$, we can express

$$
\langle \Theta(z,\bar z)\ \Theta(0,0)\rangle = \frac{H(z\bar z)}{z^2\bar z^2} \tag{49}
$$

$$
\langle T(z,\bar z)\ \Theta(0,0)\rangle = \langle \Theta(z,\bar z)\ T(0,0)\rangle = \frac{G(z\bar z)}{z^3\bar z} \tag{50}
$$

Now translational invariance implies that

$$
\partial^\mu T_{\mu z} = \partial^z T_{zz} + \partial^{\bar z} T_{\bar z z} = 2(\partial_{\bar z} T + \frac{1}{4}\partial_z \Theta) = 0 \tag{51}
$$

Inserting this into the two–point functions with $T(0,0)$ and $\Theta(0,0)$, we get

$$
\partial_{\bar z}\langle T(z,\bar z)T(0,0)\rangle + \frac{1}{4}\partial_z\langle \Theta(z,\bar z)T(0,0)\rangle = \partial_{\bar z}\Big[\frac{F(z\bar z)}{z^4}\Big] + \frac{1}{4}\partial_z\Big[\frac{G(z\bar z)}{z^3\bar z}\Big] = 0 \tag{52}
$$

$$
\partial_{\bar z}\langle T(z,\bar z)\Theta(0,0)\rangle + \frac{1}{4}\partial_z\langle \Theta(z,\bar z)\Theta(0,0)\rangle = \partial_{\bar z}\Big[\frac{G(z\bar z)}{z^3\bar z}\Big] + \frac{1}{4}\partial_z\Big[\frac{H(z\bar z)}{z^2\bar z^2}\Big] = 0 \tag{53}
$$

which yields two linear first-order ODE (with $\dot F(z\bar z) \equiv z\bar z F'(z\bar z)$ and similarly for $\dot G$ and $\dot H$):

$$
\dot F + \frac{1}{4}(\dot G - 3G) = 0 \tag{54}
$$

$$
\dot G - G + \frac{1}{4}(\dot H - 2H) = 0 \tag{55}
$$

Now if we define $C(z\bar z) \equiv 2F - G - \frac{3}{8}H$ and eliminate G from the above equations, we find that

$$
\dot C = -\frac{3}{4}H = -\frac{3}{4}(z\bar z)^2\langle \Theta(z,\bar z)\ \Theta(0,0)\rangle \le 0 \tag{56}
$$

since $\langle \Theta(z,\bar z)\ \Theta(0,0)\rangle \ge 0$ by reflection positivity. Note that $\dot C = 0$ iff $\theta(z,\bar z) \equiv 0$, which means that the system is scale invariant (i.e. at the fixed point)! Moreover, at each fixed point, $G = H \propto \Theta = 0$, hence $C = 2F = c$ is the central charge of the corresponding CFT!

The only thing left to check is how $C(z\bar{z}, \{K\}) = C(R^2, \{K\})$ transforms under RG flow, where we denote $R \equiv |z|$. Note that coarse-graining by $a \to a(1+\delta l)$ amounts to calculating the correlation functions at the new length scale $R \to R(1+\delta l)$ while keeping the couplings $\{K\}$ *fixed*:

$$\begin{aligned} C(R^2, \{K\}) &\to C(R^2(1+2\delta l), \{K\}) \\ &= C(R^2, \{K\}) + 2R^2 \delta l \ \frac{\partial}{\partial R^2} C(R^2, \{K\}) \\ &= C(R^2, \{K\}) + \delta l \ \frac{d}{dl} C(R^2, \{K\}) \end{aligned} \tag{57}$$

Hence we conclude that

$$\dot{C}(z\bar{z}, \{K\}) = (z\bar{z}) \ C'(z\bar{z}, \{K\}) \equiv R^2 \frac{\partial}{\partial R^2} C(R^2, \{K\}) = \frac{1}{2} \frac{d}{dl} C(z\bar{z}, \{K\}) \tag{58}$$

So $\dot{C}$ indeed represents how C transforms under RG flow. In summary, we have found that the function $C(z\bar{z}, \{K\})$ is monotonically decreasing along the RG flows and, at each fixed point, is stationary and equal to the central charge of the corresponding CFT. Sounds familiar? We did it!

This proof by Zamolodchikov [6] is minimalist – it simply extends the CFT machniery to describe two-point functions along RG trajectory and elegantly encode the gory calculations of RG flow by F, G, and H. Nonetheless, $c-$theorem has deep implicaitons. For example, in the tricritical Ising model, the unstable manifolds eminating from the tricritical fixed point ($c = 7/10$) can only flow to the trivial fixed points ($T = 0$ or $T = \infty$, $c = 0$) or the Ising fixed point ($c = 1/2$). Therefore, the edges of the $h \neq 0$ wings of the tricritical fixed point *must* be in the Ising universality class!

4.3 Proof via Entanglement Entropy

It turns out that we can prove the $c-$theorem with an even more minimalist argument if we simply focus on the RG flow of entanglement entropy [7]. Given any two–dimensional CFT in $\mathbb{R}_x \times \mathbb{R}_y$, we rotate one spatial dimension to imaginary time $y \to it$ and get Lorentz invariant QFT in $1+1$ dimensions. The proof relies on the following properties of entanglement entropy:

Conservation via Unitarity: $S_A = S_{A'}$ if A and A' are related by *unitary* time evolution.

Strong Subadditivity: $S_A + S_B \geq S_{A \cup B} + S_{A \cap B}$ for any two spacetime intervals A and B.

In 2D QFT Vacuum: $S_A = S_A(L_A)$ where L_A is the proper distance of A.

In 2D CFT Vacuum: $S_A = \frac{c}{3} \log(L_A/a_{UV})$, where a_{UV} is the UV–cutoff in *spacetime*.

Proof. We need the help of our good old friends Alice and Bob. At $t = 0$ in the rest frame F, Alice can measure the entropy S_A of the subsystem A and Bob can measure S_B, where for fairness of labor we have $L_A = L_B$. Now let's boost Alice and Bob onto their merry ways to frames F_A and F_B with equal but opposite velocities $\pm v$ with respect to F, as shown in Figure 1 below.

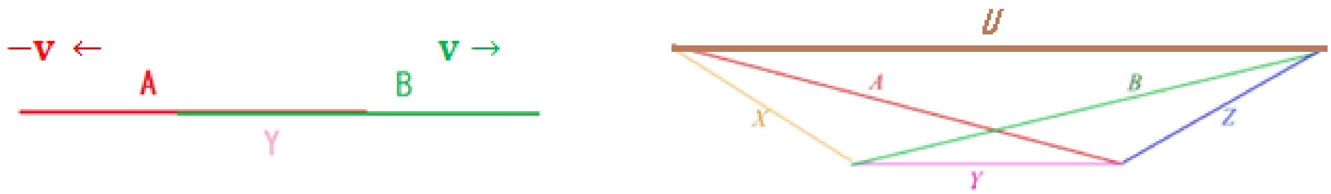

Figure 1: Left: A and B in the same rest frame; Right: A and B in their boosted frames;

What can Alice measure in F_A? Well, if we draw the causal diamonds, we find that A as seen from its boosted frame F_A is causally dependent upon $X \cup Y$. Hence if we know everything about the systems $X \cup Y$, we can close our eyes, let time evolve, and get everything in Alice's boosted frame for free! No additional entropy is induced since unitarity ensures probability conservation!

Running through the same argument for Bob, we find that

$$S_A = S_{X\cup Y} \qquad S_B = S_{Y\cup Z} \qquad S_Y = S_{A\cap B} \qquad S_U = S_{A\cup B} = S_{X\cup Y\cup Z} \tag{59}$$

Note that strong subadditivity (SSA) says that

$$S_A + S_B \geq S_Y + S_{X\cup Y\cup Z} = S_Y + S_U \tag{60}$$

Since the entropy depends only on the proper length, we can parametrize the lengths by $L_Y = R$ and $L_U = r$ with $R < r$. After some algebra we get $L_A = L_B = \sqrt{Rr}$. Hence SSA implies that

$$2S(\sqrt{rR}) \geq S(r) + S(R) \tag{61}$$

Now if we take $r = R + \epsilon$ with $\epsilon \ll R$ and expand the above inequality to $\mathcal{O}(\epsilon^2)$:

$$\begin{aligned} 2S(\sqrt{R^2+\epsilon R}) &\approx 2S(R + \frac{\epsilon}{2} - \frac{\epsilon^2}{8R}) \\ &\approx 2S(R) + \epsilon S'(R) + \frac{1}{4}\epsilon^2 S''(R) - \frac{\epsilon^2}{4R}S'(R) \\ &\geq S(R+\epsilon) + S(R) \\ &\approx 2S(R) + \epsilon S'(R) + \frac{1}{2}\epsilon^2 S''(R) \end{aligned} \tag{62}$$

we get the inequality of the new function $C(R) \equiv 3RS'(R)$:

$$RS''(R) + S'(R) = \frac{d}{dR}(RS'(R)) \equiv \frac{1}{3}\frac{d}{dR}C(R) \leq 0 \tag{63}$$

So what is $C(R)$? Recall that at each fixed point, the CFT entanglement entropy is given by

$$S(R) = \frac{c}{3}\log(\frac{R}{a_{UV}}) \tag{64}$$

Therefore, $C(R) = 3RS'(R) = c$ is equal to the central charge at each fixed point! If we go back to the original 2D CFT by $it \to y$, we can interpret change of the proper distance $R \to r = R + \epsilon$ as the spatial RG flow! To recap, we found a monotonically decreasing function $C(R)$ along the RG trajectory which is stationary and equal to the central charge of the corresponding CFT at each fixed point. Sounds familiar? Thanks to the generous help of Alice and Bob, we did it again! This time, we only used Lorentz invariance, unitarity, and the entropy formula (which actually requires a rather technical derivation from the CFT machineary we have developed so far, see [1] for details).

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