An Introduction to the BKT Phase Transition

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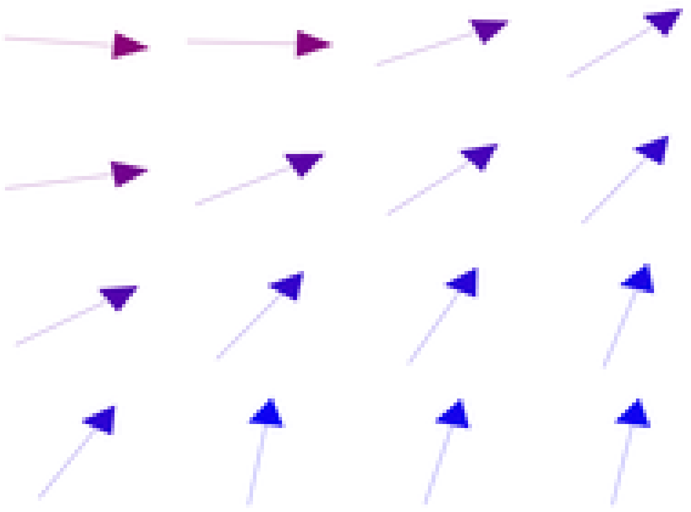
PHYS 7653 – Statistical Mechanics II

The 2-D XY Model

A system of continuous spins in a 2 dimensional lattice, with only nearest neighbor interactions between the spins.

For purposes of today's treatment:

- The lattice is square, with length L, lattice spacing a.
- The Hamiltonian governing the system is shown on the right.



$$-\beta H = K \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

Is there a phase transition?

- **Mermin–Wagner theorem** : Continuous symmetries cannot be **spontaneously broken** at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leq 2$.
- Intuitively, this means that long-range fluctuations can be created with little energy cost and since they increase the entropy they are favored. **(See Section 2.5 of the pre-class reading)**

So, naively, the $2D$ XY model should have no phase transition!

The XY model at low temperatures

For low temperatures, we can think of θ as a field and write the Hamiltonian as:

$$-\beta H = \frac{K}{2} \int \mathrm{d}^2\mathbf{x}\ (\nabla\theta)^2$$

$$\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{x}) \rangle = \mathrm{Re}\left\langle e^{i(\theta(0)-\theta(\mathbf{x}))} \right\rangle = \mathrm{Re}\left[e^{-\langle(\theta(0)-\theta(\mathbf{x}))^2\rangle/2} \right].$$

In section 2.5 we saw that in two-dimensions Gaussian fluctuations grow logarithmically $\langle(\theta(0) - \theta(\mathbf{x}))^2\rangle/2 = \ln(|\mathbf{x}|/a)/2\pi K$, where a denotes a short distance cut-off (i.e. lattice spacing). Therefore, at low temperatures the spin-spin correlation function decays *algebraically* as opposed to exponential.

$$\boxed{\langle \mathbf{S}(0) \cdot \mathbf{S}(\mathbf{x}) \rangle \simeq \left(\frac{a}{|\mathbf{x}|} \right)^{1/2\pi K}}$$

Power Law Decay

How about high temperatures?

1. Now you can expand the Hamiltonian in powers of K, since the coupling between spins can be assumed to be weak in high temperatures. *Expand upto* $\mathcal{O}(K)$.

2. *Write the partition function in this approximation:*

$$\mathcal{Z} = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cdots$$

3. *Using* $\mathcal{Z}$, *evaluate* $\langle \mathbf{S_0}.\mathbf{S_x} \rangle$

Hints:

$$\int_0^{2\pi} \frac{d\theta_i}{2\pi} \cos(\theta_i - \theta_j) = 0$$

$$\int_0^{2\pi} \frac{d\theta_k}{2\pi} \cos(\theta_i - \theta_k) \cos(\theta_k - \theta_j) = \frac{1}{2} \cos(\theta_i - \theta_j)$$

How about high temperatures?

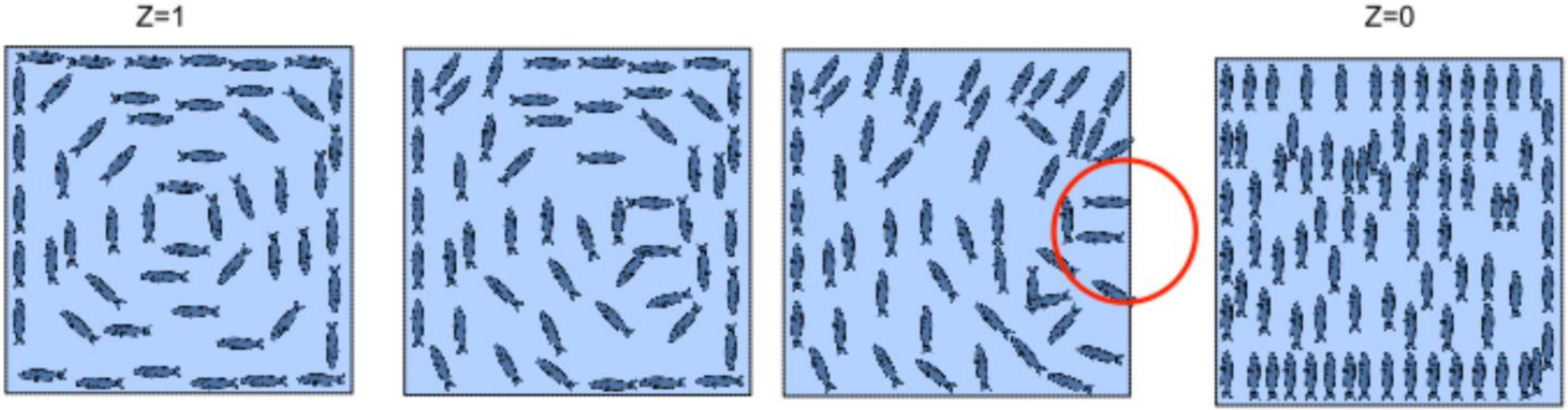
Answers:

$$\mathcal{Z} = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} e^{-\beta H} = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + K\cos(\theta_i - \theta_j) + O(K^2)\right]$$

$$\langle \mathbf{S}_0 \cdot \mathbf{S}_\mathbf{x} \rangle \sim \left(\frac{K}{2}\right)^{|\mathbf{x}|} \sim \exp\left[-|\mathbf{x}|/\xi\right]$$

Exponential Decay!

Hence Phase Transition?



No continuous deformations possible

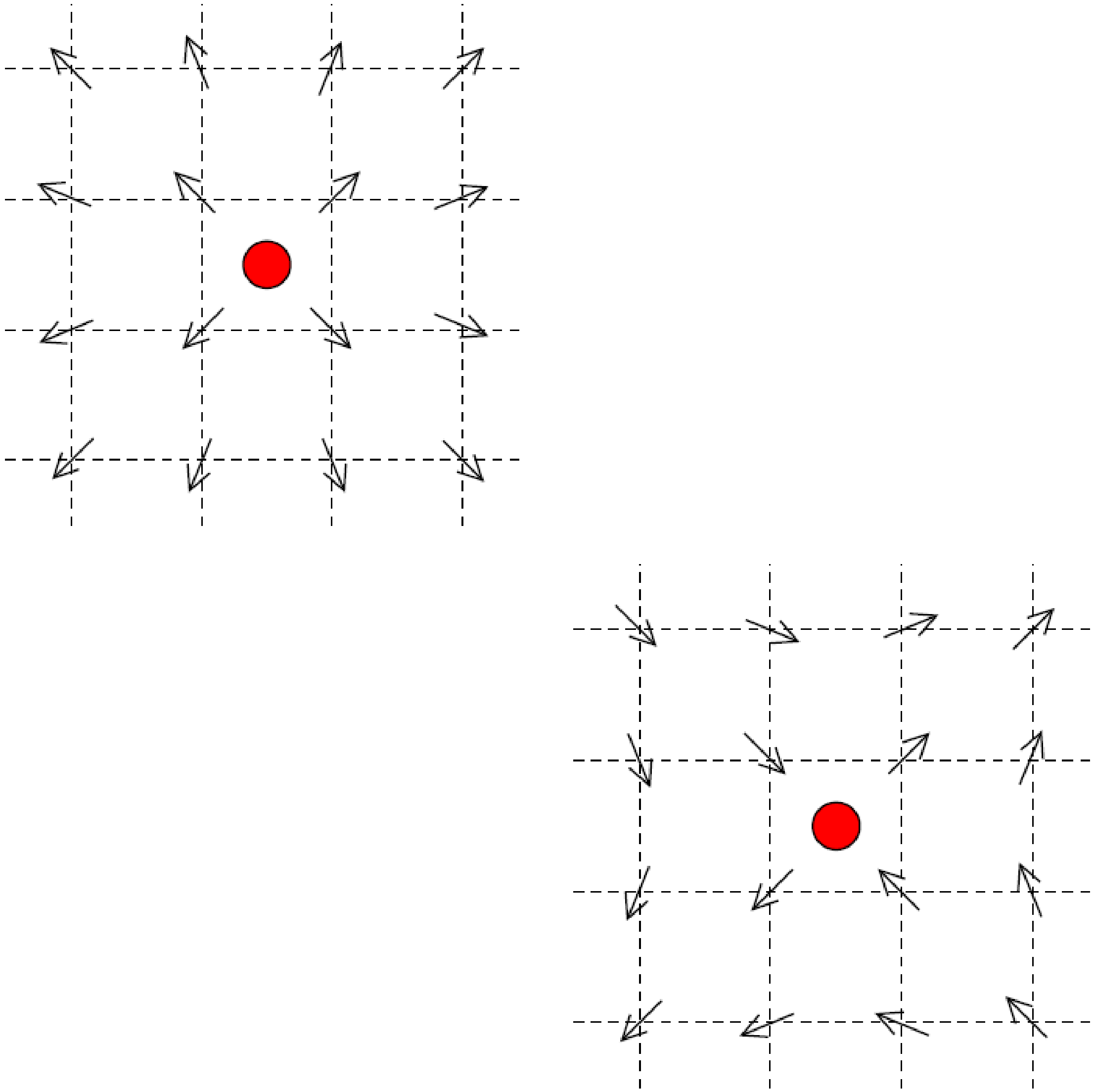
Vortices:

We talked about vortices and what they are in the pre-class exercise.

$$\boldsymbol{v} = \boldsymbol{\nabla}\theta$$

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 2\pi\delta^2(\boldsymbol{x})$$

$$\boldsymbol{\nabla}.\boldsymbol{v} = 0$$



Energy of this configuration

$$\oint \nabla\theta \cdot d\ell = 2\pi n \quad \Longrightarrow \quad \nabla\theta = \frac{n}{r}\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z$$

The energy cost from a single vortex of charge n has contributions from the core region, as well as from the relatively uniform distortions away from the centre. The distinction between regions inside and outside the core is arbitrary, and for simplicity, we shall use a circle of radius a to distinguish the two, i.e.

$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d\mathbf{x}(\nabla\theta)^2$$

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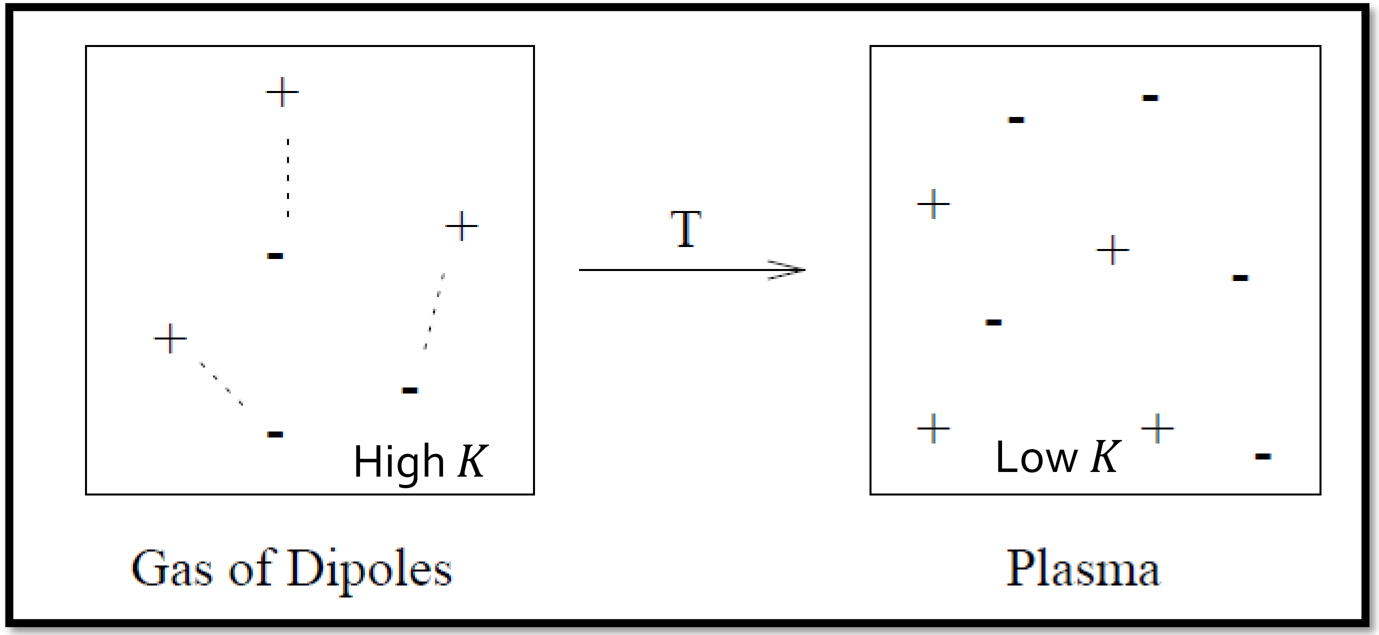
$$\beta E_n = \beta E_n^0(a) + \frac{K}{2}\int_a d\mathbf{x}(\nabla\theta)^2$$

$$= \beta E_n^0(a) + \pi K n^2 \ln\left(\frac{L}{a}\right)$$

A naïve analysis of the one vortex partition function

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right]$$



$$\nabla\theta = \nabla\theta_+ + \nabla\theta_- \approx 2\mathbf{d}\cdot\nabla\left(\frac{\hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_z}{|\mathbf{r}|}\right)$$

Coulomb Gas?

- We talked about **topological charges** – we saw that they behave somewhat like electric charges since we saw that:

$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} . \vec{E} = 2\pi n \delta^2(\vec{x})$$

Where $E_x = v_y$ and $E_y = -v_x$

We can therefore define: $\vec{E} = -\nabla\chi$, for a scalar χ

After a redefinition $K \to \frac{1}{g^2}$, we would have:

$$F = \int d^2\boldsymbol{x} \, \frac{1}{2g^2} (\nabla\chi)^2$$

The Free Energy and the Partition Function

$$F = \int d^2x \; -\frac{1}{2g^2}\chi \nabla \cdot \mathbf{E} = \frac{\pi}{g^2}\sum_{i \neq j} n_i n_j \log\left(\frac{|\mathbf{X}_i - \mathbf{X}_j|}{a}\right) + \sum_i n_i^2 F_{\text{core}}$$

We assume a dilute gas model, where defects of the same charge repel each other strongly and are typically far apart, and defects of opposite charges that are close together annihilate.

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \exp\left(\frac{\pi}{g^2}\sum_{\alpha \neq \beta} V_\alpha V_\beta \log\left(\frac{|\mathbf{X}_\alpha - \mathbf{X}_\beta|}{a}\right) - \sum_\alpha V_\alpha^2 F_{\text{core}}\right)$$

Where $V_\alpha = -1,0,1$

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \exp\left(\frac{\pi}{g^2}\sum_{\alpha\neq\beta} V_\alpha V_\beta \log\left(\frac{|\mathbf{X}_\alpha - \mathbf{X}_\beta|}{a}\right) - \sum_\alpha V_\alpha^2 F_{\text{core}}\right)$$

$$\int \mathcal{D}\phi \; \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2 + f(\mathbf{x})\phi(\mathbf{x})\right) \sim \exp\left(-\frac{1}{4\pi}\int d^2x d^2y \; f(\mathbf{x}) \log|\mathbf{x}-\mathbf{y}| f(\mathbf{y})\right)$$

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \int \mathcal{D}\phi \; \exp\left(-\int d^2x\, \frac{1}{2}(\nabla\phi)^2 + \sum_\alpha \frac{2\pi i}{g} V_\alpha \phi_\alpha - V_\alpha^2 F_{\text{core}}\right)$$

A Sneaky trick!

"Sine-Gordon"-ization!

$$Z_{\text{vortex}} \sim \sum_{\{V_\alpha\}} \int \mathcal{D}\phi \ \exp\left(-\int d^2x \ \frac{1}{2}(\nabla\phi)^2 + \sum_\alpha \frac{2\pi i}{g} V_\alpha \phi_\alpha - V_\alpha^2 F_{\text{core}} \right)$$

$$\begin{aligned}
Z_{\text{vortex}} &= \int \mathcal{D}\phi \ \exp\left(-\frac{1}{2}\int d^2x \ (\nabla\phi)^2\right) \prod_\alpha \sum_{V_\alpha=-1,0,+1} e^{\frac{2\pi i}{g}V_\alpha\phi_\alpha + V_\alpha^2 F_{\text{core}}} \\
&= \int \mathcal{D}\phi \ \exp\left(-\frac{1}{2}\int d^2x \ (\nabla\phi)^2\right) \prod_\alpha \left[1 + 2e^{-F_{\text{core}}} \cos(2\pi\phi_\alpha/g)\right] \\
&\approx \int \mathcal{D}\phi \exp\left(-\frac{1}{2}\int d^2x \ (\nabla\phi)^2 + \frac{2}{a^2}e^{-F_{\text{core}}} \int d^2x \ \cos\left(\frac{2\pi\phi}{g}\right)\right)
\end{aligned}$$

The "Sine Gordon" Model

$$F = \int d^2\mathbf{x} \left[\frac{1}{2}(\nabla\phi)^2 - \lambda\cos(\beta\phi) \right]$$

$$\lambda = \frac{2}{a^2} e^{-F_{\mathrm{core}}}$$

$$\beta = \frac{2\pi}{g}$$

- We treat the cosine term as a perturbation to the gradient part of the Hamiltonian in order to perform our RG operations on the coupling constants.

Momentum Space perturbative RG

We want to integrate out momenta in a shell given by:

$$\Lambda/\zeta < k < \Lambda$$

We have

$$\beta H[\phi] = \beta H_0[\phi] + \mathcal{U}[\phi]$$

where

$$\phi = \phi_>(x) + \phi_<(x)$$

A calculation of the partition function upto first order in λ allows us to write:

$$\beta H_< = \beta H_0[\phi_<] + \langle \mathcal{U} \rangle_> + \text{const.}$$

$$\langle \mathcal{U} \rangle_> = \lambda \langle \cos(\phi_< + \phi_>) \rangle_> = \lambda \zeta^{-\beta^2/4\pi} \cos(\beta \phi_<)$$

$$\begin{aligned} \langle g \cos[\lambda(\phi_<(\boldsymbol{x}) + \phi_>(\boldsymbol{x}))] \rangle_> &= g \mathrm{Re}\left[e^{i\lambda\phi_<(\boldsymbol{x})} \langle e^{i\lambda\phi_>(\boldsymbol{x})} \rangle_> \right] \\ &= g e^{-\lambda^2 \langle \phi_>^2(\boldsymbol{x}) \rangle / 2} \cos(\lambda \phi_<(\boldsymbol{x})) \end{aligned}$$

Momentum Space perturbative RG

Therefore, as a result of the coarse graining,

$$F[\phi_<] = \int \mathrm{d}^2\mathbf{x} \left[\frac{1}{2}(\nabla\phi_<)^2 - \lambda\zeta^{-\beta^2/4\pi}\cos(\beta\phi_<) \right]$$

Now $\mathbf{x} \to \mathbf{x}' = \mathbf{x}/\zeta$, and so:

$$F[\phi_<] = \int \mathrm{d}^2\mathbf{x}'\zeta^2 \left[\frac{1}{2}(\nabla\phi_<)^2 - \lambda\zeta^{-\beta^2/4\pi}\cos(\beta\phi_<) \right]$$

This gives us: $\boxed{\lambda(\zeta) = \lambda_0\zeta^{(2-\beta^2/4\pi)}}$

So λ is irrelevant for $g^2 < \frac{\pi}{2}$ and relevant otherwise!

Conclusions from first order RG:

$g^2 < \frac{\pi}{2}$	$g^2 > \frac{\pi}{2}$
• λ is irrelevant • Upon subsequent RG transformations, the free energy is dominated by the gradient terms • As we have seen before, this implies a **power law fall-off** of correlations	• λ is now relevant • Upon subsequent RG, the free energy has increasing cosine contribution. • Expanding about the vacuum, will give terms like ϕ^2 which lead to **exponential fall-off** of correlations

A cross-check:

$$F_{\text{vortex}} = \frac{1}{2g^2}\int d^2x\, (\nabla\theta)^2 = \frac{\pi n^2}{g^2}\log\left(\frac{L}{a}\right) + F_{\text{core}}$$

$$\mathcal{Z}_1(n) \approx \left(\frac{L}{a}\right)^2 \exp\left[-\beta E_n^0(a) - \pi K n^2 \ln\left(\frac{L}{a}\right)\right]$$

$$p(\text{vortex}) = \left(\frac{L}{a}\right)^2 \frac{e^{-F_{\text{vortex}}}}{Z} = \frac{e^{-F_{\text{core}}}}{Z}\left(\frac{L}{a}\right)^{2-\pi/g^2}$$

We see that, when g^2 surpasses a critical value,

$$g^2 > g_{KT}^2 = \frac{\pi}{2}$$

then there is no suppression of vortices; their entropy, coming from the fact that they can sit anywhere on the plane, wins out over their energetic cost. As in the previous section, g^2 can be viewed as the temperature of the system, and g_{KT}^2 translates into a temperature scale T_{KT}, above which vortices proliferate.