Kosterlitz-Thouless universality family: From the XY model to a grown network

David Hathcock

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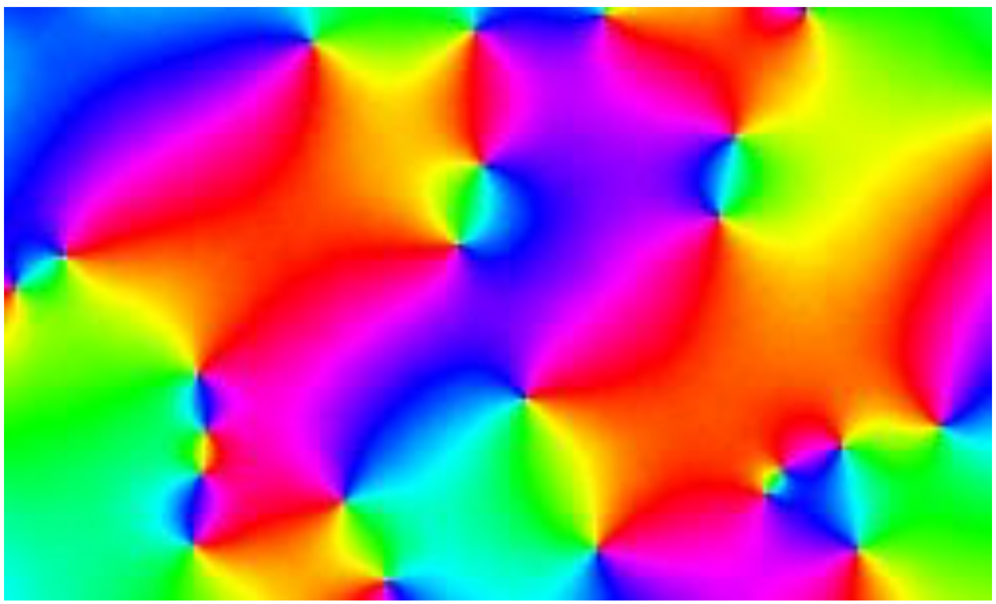
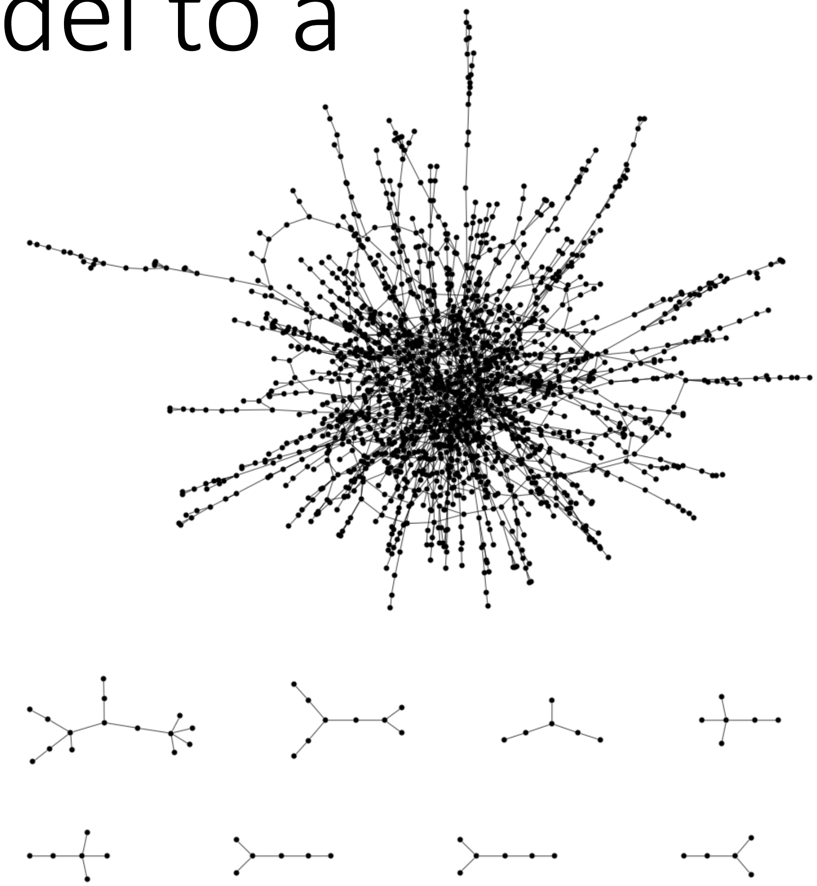


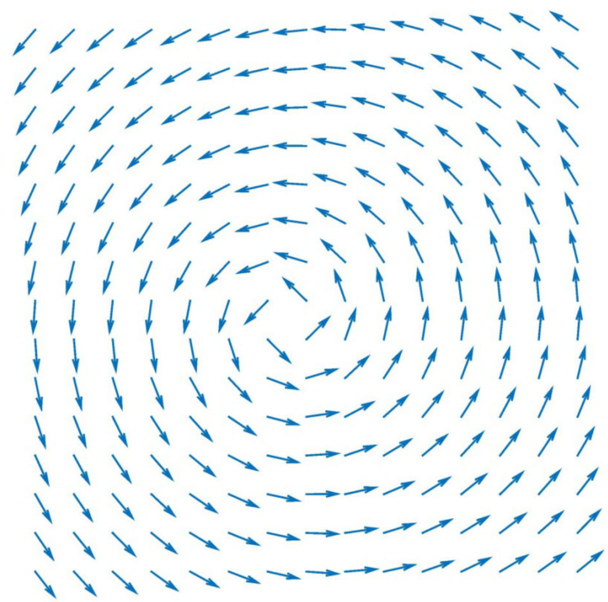


Review: Topological Charges

- In the XY model, we see the emergence of topological charges with 2D Coulomb interaction:

$$S[\theta] = -K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

$$\rightarrow S_n = S_n^{core}(a) + \pi K n^2 \log(L/a)$$



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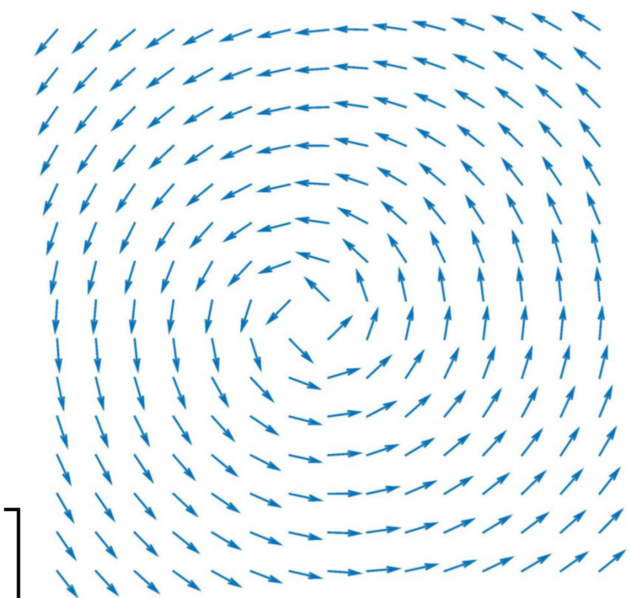

- Single Vortex Partition Function:

$$\mathcal{Z} = \left(\frac{L}{a}\right)^2 \exp\left[- S_1^{core}(a) - \pi K \log\left(\frac{L}{a}\right)\right]$$

Review: Topological Charges

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$K > 2/\pi \rightarrow$ energy dominant, no vortices

$K < 2/\pi \rightarrow$ entropic contribution dominates, formation of vortices

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- N – Vortex Coulomb Gas:

$$\mathcal{Z}_v = \sum_{N=0}^{\infty} \frac{y_0^{2N}}{(N!)^2} \int \left(\prod_{i=1}^{2n} d^2 r_i\right) \exp\left[\pi K \sum_{i<j} n_i n_j \log|\mathbf{r}_i - \mathbf{r}_j|\right]$$

$y_0 = e^{-S_{\pm 1}^{core}(a)}$ – "Fugacity": Energy cost of vortex core

$K n_i n_j \log|\mathbf{r}_i - \mathbf{r}_j|$ – cost of interactions, deformations near charges

Physical Interpretation

- How two distant topological charges interact:
  - Low Temperature – Coulomb's law
  - High Temperature – Screening fluctuations in spins, newly formed intermediate charges
  - High fugacity – Increases probability of new charges forming
- These two mechanisms contribute to phase transition
  - Insulating phase becomes conducting (external charges completely screened)

Renormalization Group – Sine-Gordon Model

$$F[\phi] = \int d^2x \; \frac{1}{2}(\nabla\phi)^2 - \frac{\alpha}{a^2\beta^2}\cos(\beta\phi)$$

$$\beta = 2\pi\sqrt{K}$$

$$\alpha/\beta^2 = 2y_0$$

Renormalization Procedure:

Integrate out high momentum modes: $\phi_{\mathbf{k}} = \phi_{\mathbf{k}}^- + \phi_{\mathbf{k}}^+, \quad \Lambda/\zeta < k < \Lambda$

Effective free energy: $F'[\phi^-] = F_0[\phi^-] - \log\left\langle e^{-F_{\rm int}[\phi^- + \phi^+]}\right\rangle$

Renormalization – Perturbations in α

- Second order in α necessary for full RG flow equation

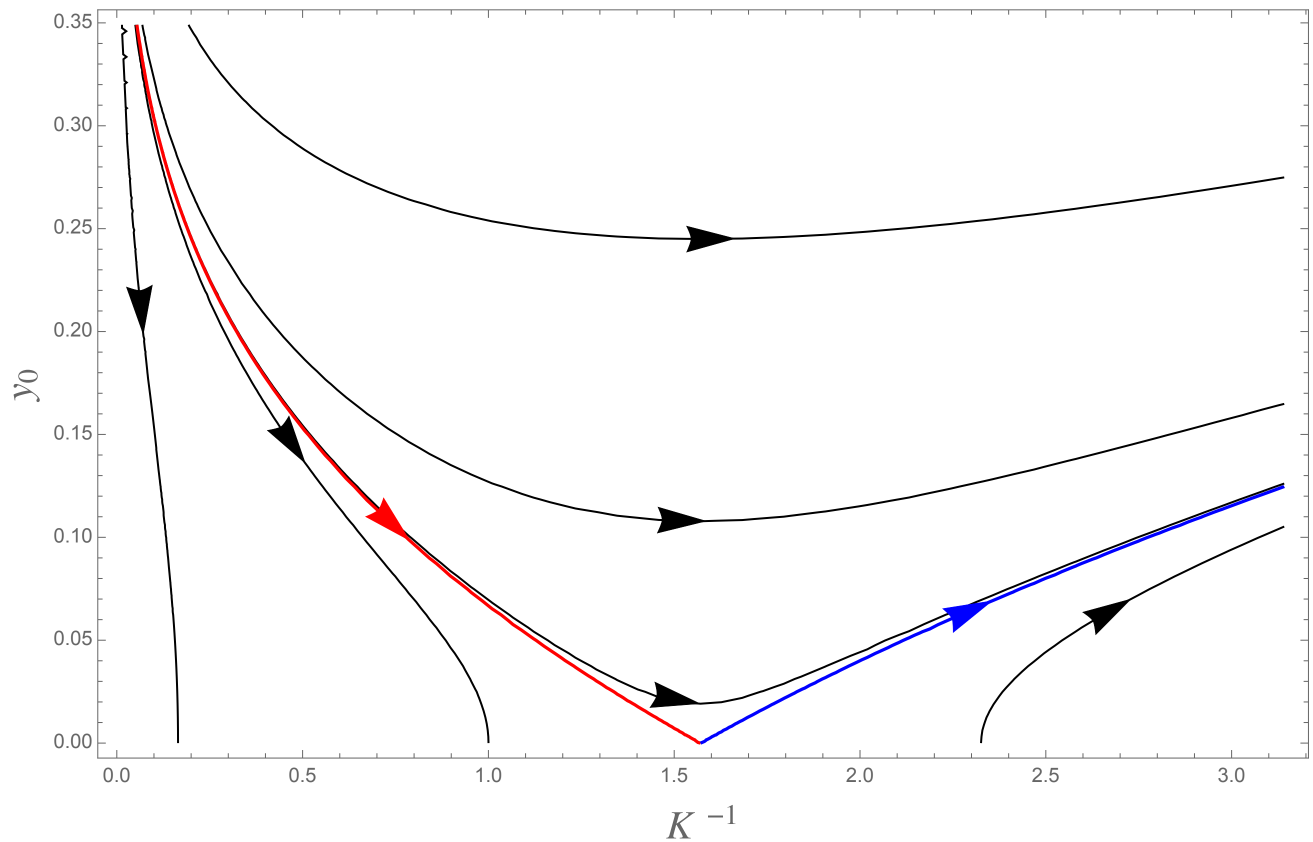
$$\left\langle \exp\left(\frac{\alpha}{a^2\beta^2}\int d^2x\cos(\beta(\phi^- + \phi^+))\right)\right\rangle_+ \approx 1 + \frac{\alpha}{a^2\beta^2}\int d^2x \left\langle \cos(\beta(\phi^- + \phi^+))\right\rangle_+ + \left(\frac{\alpha}{a^2\beta^2}\right)^2 \int\int d^2x d^2y \left\langle \cos(\beta(\phi^-_{\mathbf{x}} + \phi^+_{\mathbf{x}}))\cos(\beta(\phi^-_{\mathbf{y}} + \phi^+_{\mathbf{y}}))\right\rangle_+$$

- The RG flow equations have been calculated in the literature to third order in α (the 4^{th} order term vanishes)

Flow Equations – Lowest Order

$$\frac{dK^{-1}}{d\ell} = 4\pi^3 y_0^2$$

$$\frac{dy_0}{d\ell} = (2 - \pi K) y_0$$



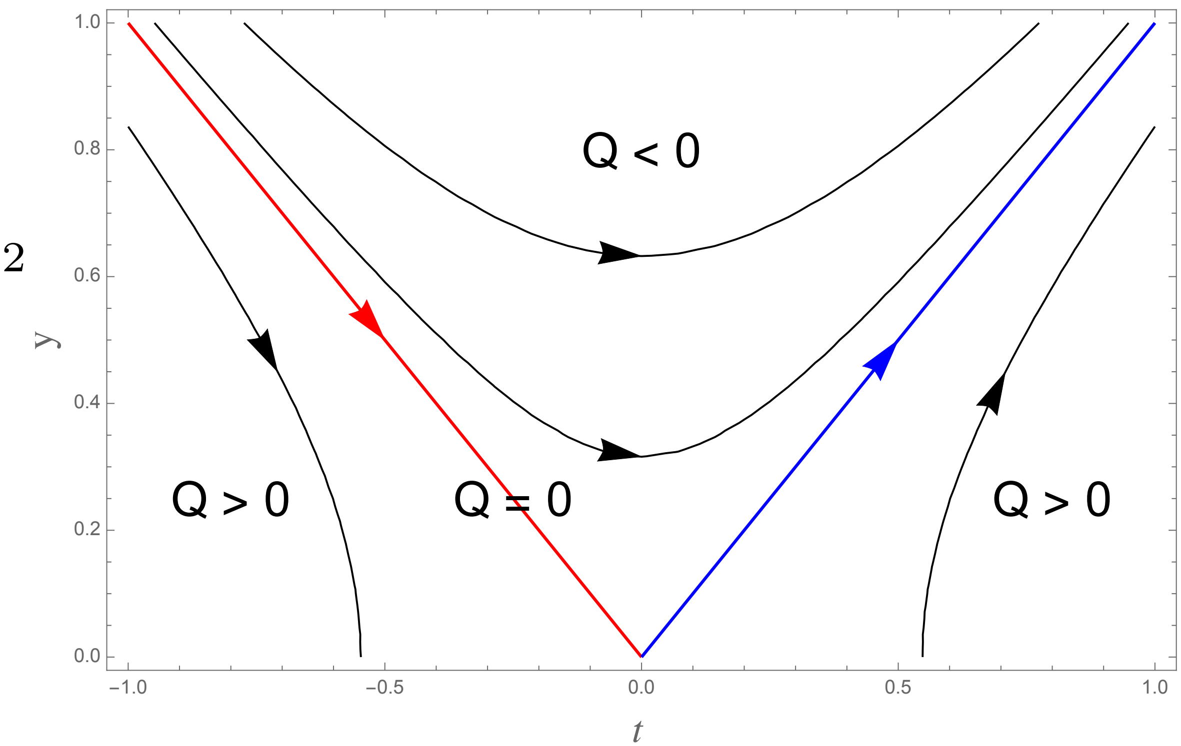
Near the Fixed Point

Let $t = K^{-1} - \pi/2,$ expanding around the fixed point $(t, y) = 0:$

$$\frac{dt}{d\ell} = y^2$$

$$\frac{dy}{d\ell} = ty$$

Invariant function: $Q = t^2 - y^2$



Exponential Divergent Correlation Length

- For $T>T_c$ the correlation length diverges exponentially

$$\xi \sim e^{\pi/\sqrt{Q}}$$

$$\frac{dt}{d\ell} = y^2$$

$$\frac{dy}{d\ell} = ty$$

$$Q = t^2 - y^2$$

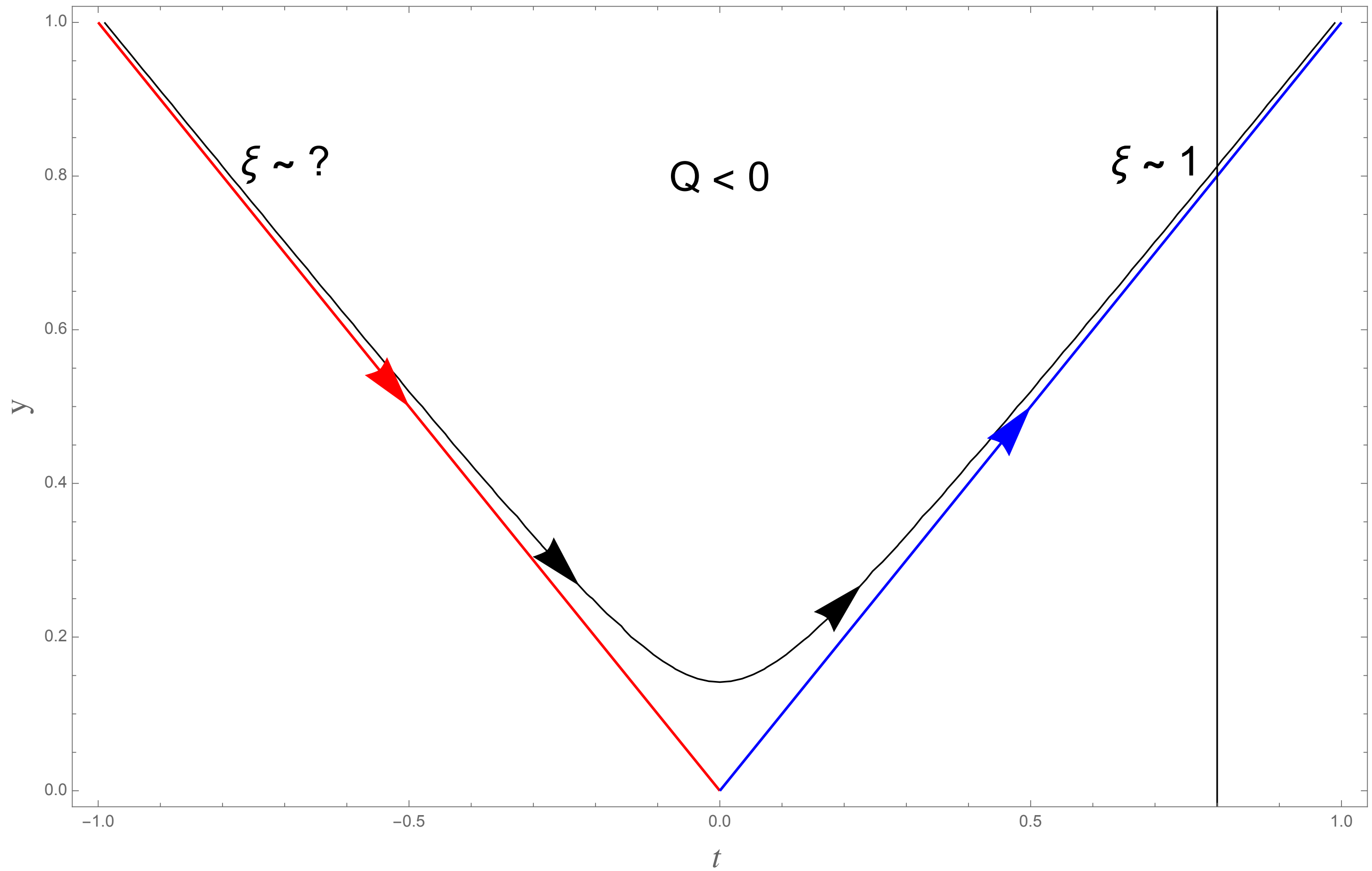
- To see this, we may estimate the time necessary to coarse grain until the correlation length is 1

Correlation Length Scaling Form

$$\frac{dt}{d\ell} = y^2$$

$$\frac{dy}{d\ell} = ty$$

$$Q = t^2 - y^2$$

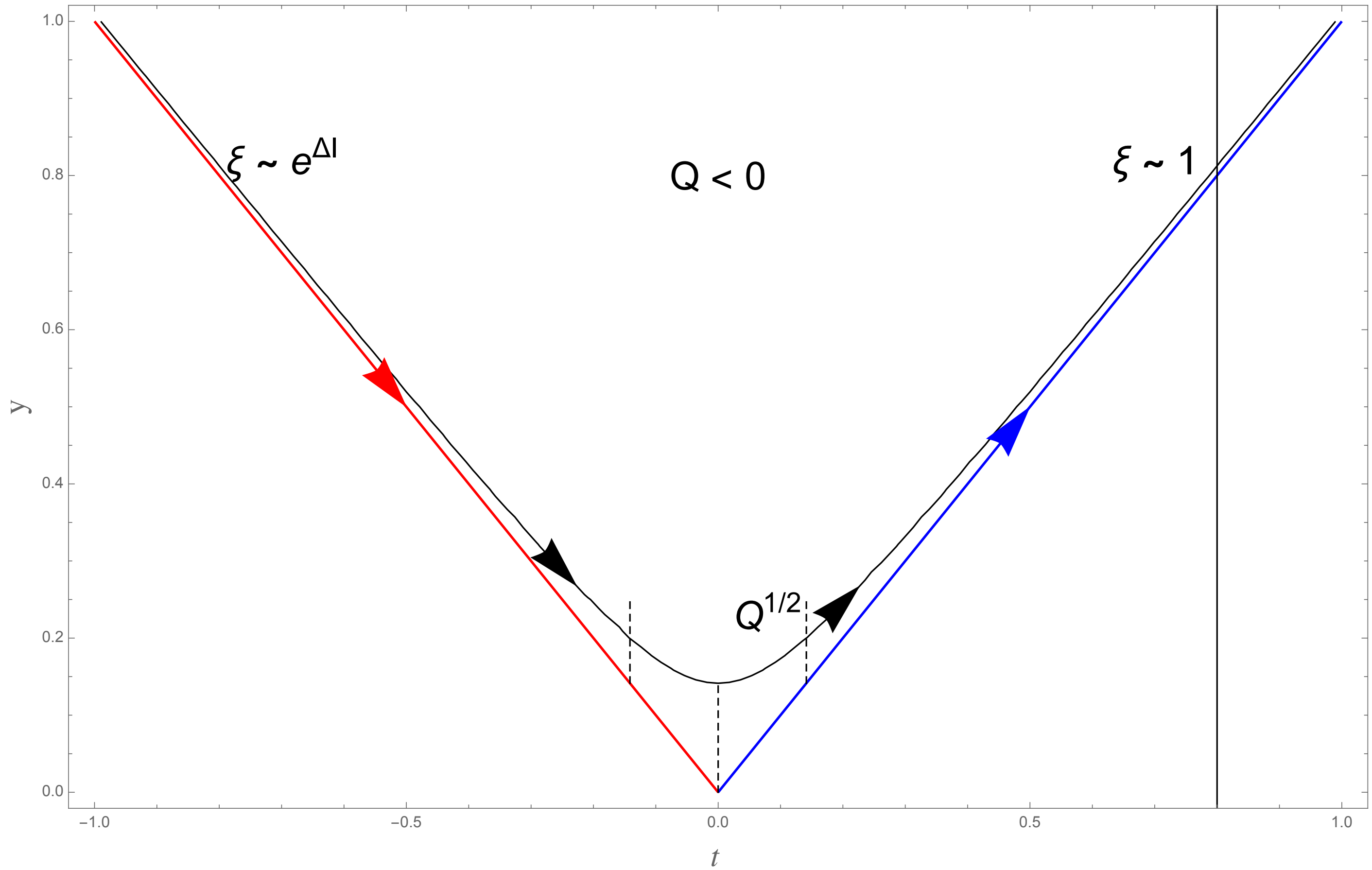


Correlation Length Scaling Form

$$\frac{dt}{d\ell} = y^2 \sim |Q|$$

$$\frac{dt}{d\ell}\Delta\ell \sim \sqrt{|Q|}$$

$$\xi \sim e^{\Delta\ell} = e^{b/\sqrt{|Q|}}$$



Higher Order Corrections & Normal Form

- The flow equations have been computed to 3rd order using the Sine-Gordon model

$$\frac{d\alpha}{d\ell} = -2\alpha\delta - \frac{5}{64}\alpha^3$$
$$\frac{d\delta}{d\ell} = -\frac{1}{32}\alpha\delta + \frac{1}{16}\alpha\delta, \quad \delta = \beta^2/8\pi - 1$$

- By an analytic redefinition of scaling variables,

$$\frac{dt}{d\ell} = y^2 - b_0 y^2 t, \quad \text{with } b_0 = 3/2 \text{ universal}$$
$$\frac{dy}{d\ell} = ty$$

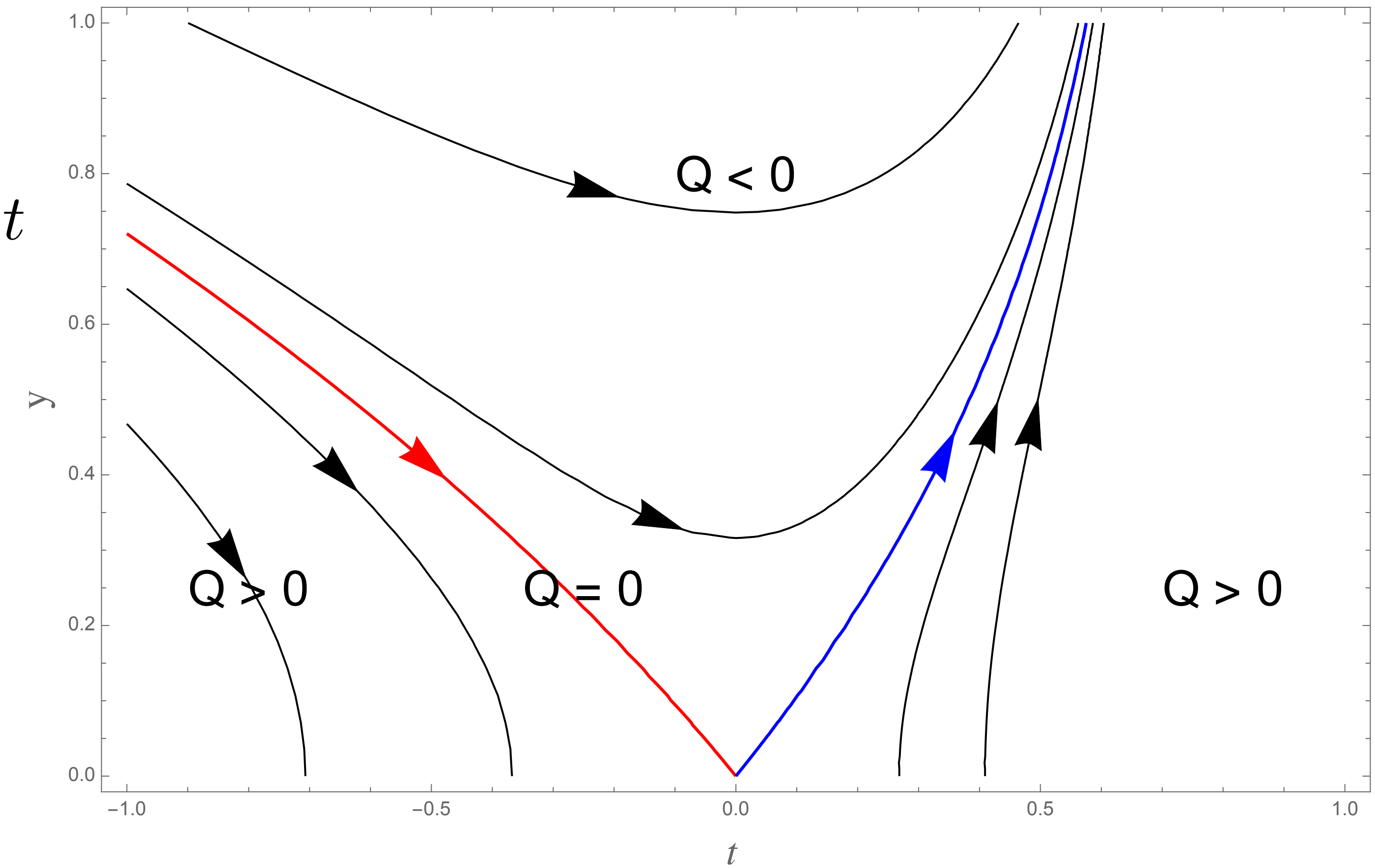
Modified RG flow diagram

Invariant function:

$$t^2 - y^2 + t^3 + 9/8t^4 + \mathcal{O}(t^5)$$

$$\frac{dt}{d\ell} = y^2 - 3/2y^2 t$$

$$\frac{dy}{d\ell} = ty$$



General Normal Form

$$\frac{dt}{d\ell} = y^2(1 - tf(t^2))$$
$$\frac{dy}{d\ell} = ty$$

- f(x) is a universal function, i.e. the coefficients in its power series are universal

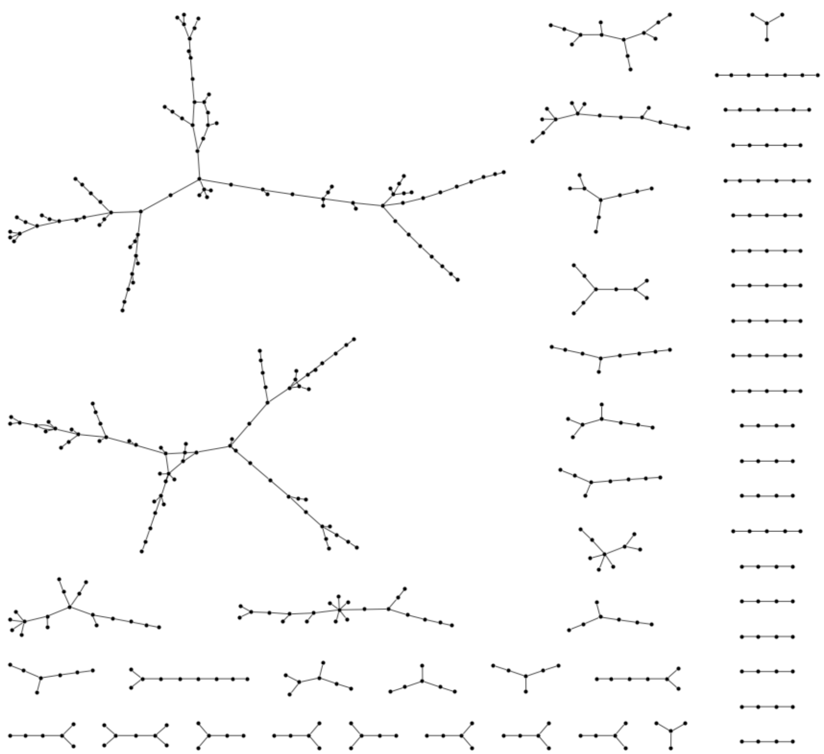
$$\text{Invariant function: } Q = 2\int_0^t \frac{sds}{1 - sf(s^2)} - y^2$$

Why is the Normal form useful?

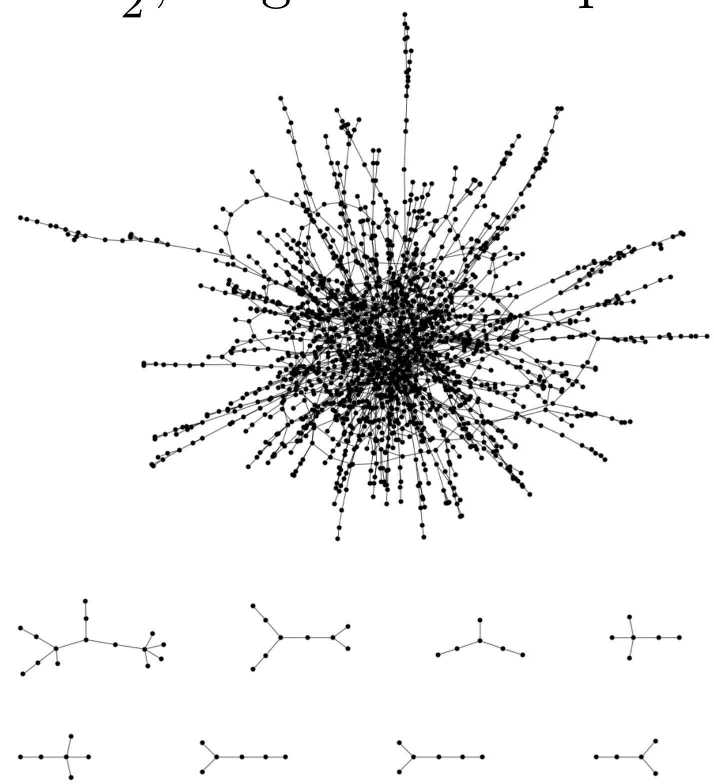
- Allows us to distinguish between different models and characterize universality classes
- Universality families group different classes by bifurcation
- Provides testable scaling predictions, critical exponents
- Can help characterize finite size scaling (ex. 4D Ising model)

Grown Network

$\delta = \frac{3}{16}$, largest 50 components



$\delta = \frac{1}{2}$, largest 10 components

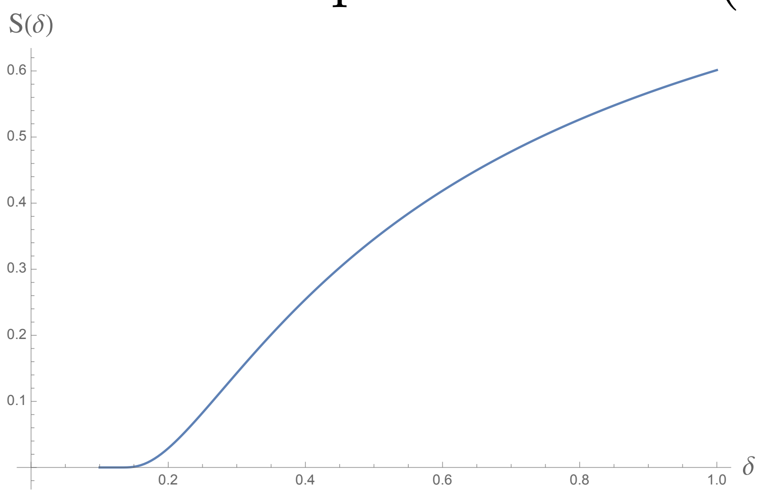


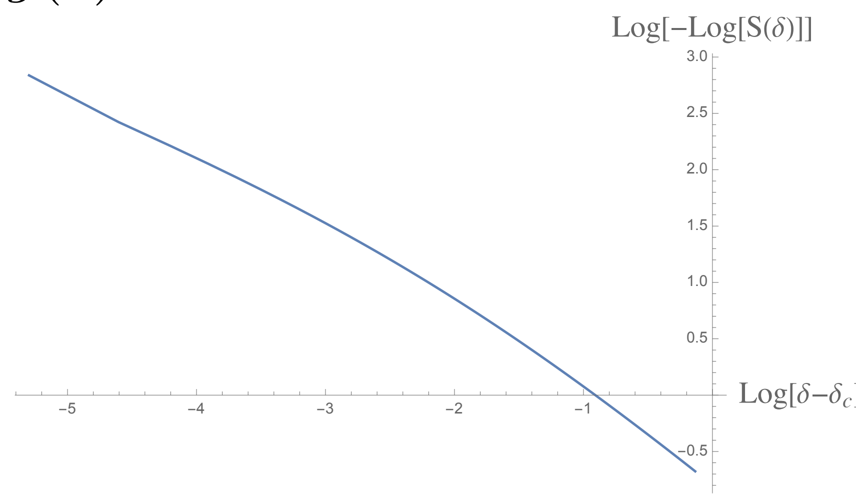
Generating Function and Giant Component

$$g(x) = \sum_{k=1}^{\infty} b_k x^k \qquad\qquad g'(x) = \frac{1}{2\delta}\frac{1 - g(x)/x}{1 - g(x)}$$

$g(1)$ – Probability of random vertex being in finite component

Giant component size: $S(\delta) = 1 - g(1) \sim e^{\alpha(\delta - \delta_c)^{-1/2}}$



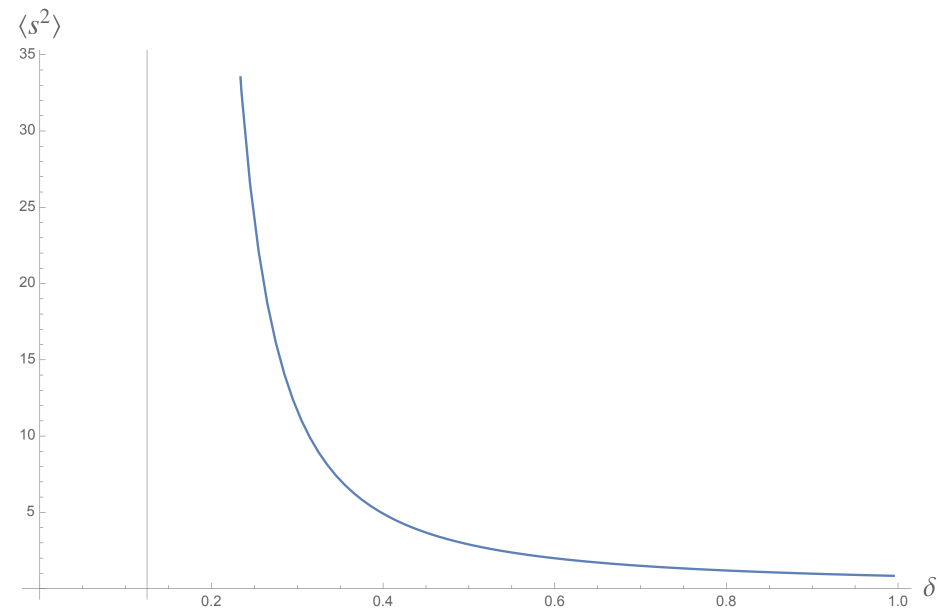


Component size moments

$$\langle s(s-1)(s-2)\cdots(s-k)\rangle = g^{(k+1)}(1)/g(1)$$

$$\langle s^2\rangle \sim g''(1)/g(1) \sim \frac{1}{2\delta S(\delta)} \sim e^{-\alpha(\delta-\delta_c)^{-1/2}}$$

Conjecture: $\langle s^k\rangle \sim \dfrac{1}{\delta^{k-1}S(\delta)^{2k-2}}$

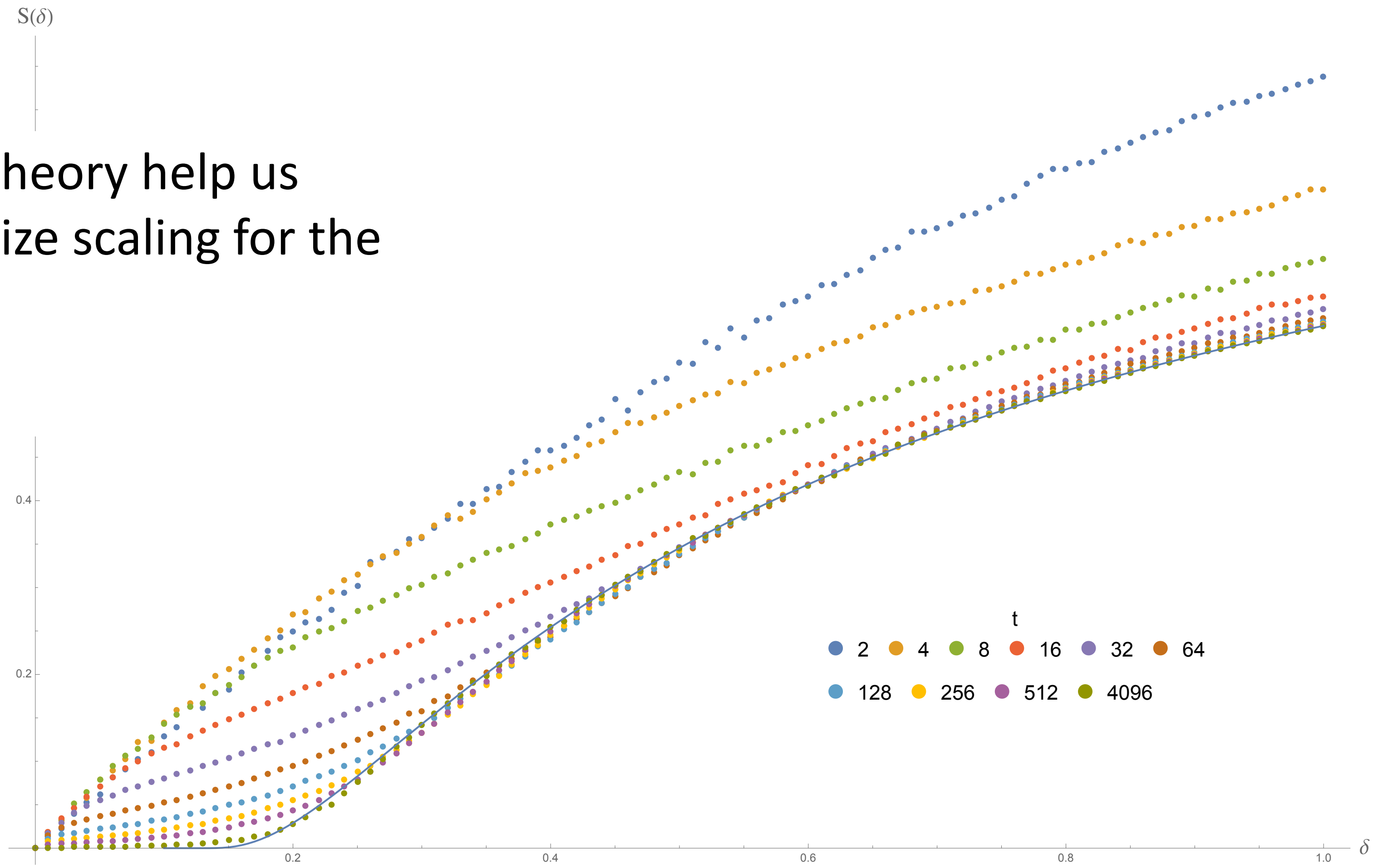


XY – Network correspondence

- We see very similar scaling forms to the XY model in the network
- No obvious mapping between the two systems (unlike Coulomb gas, Sine-Gordon)
- Are the models the same universality class? Or simply both in the Kosterlitz-Thouless universality family?
- Primary challenges:
  - Only one parameter in the network model. Can’t perform scaling collapses by varying a second parameter
  - How do we write down the invariant scaling combination?

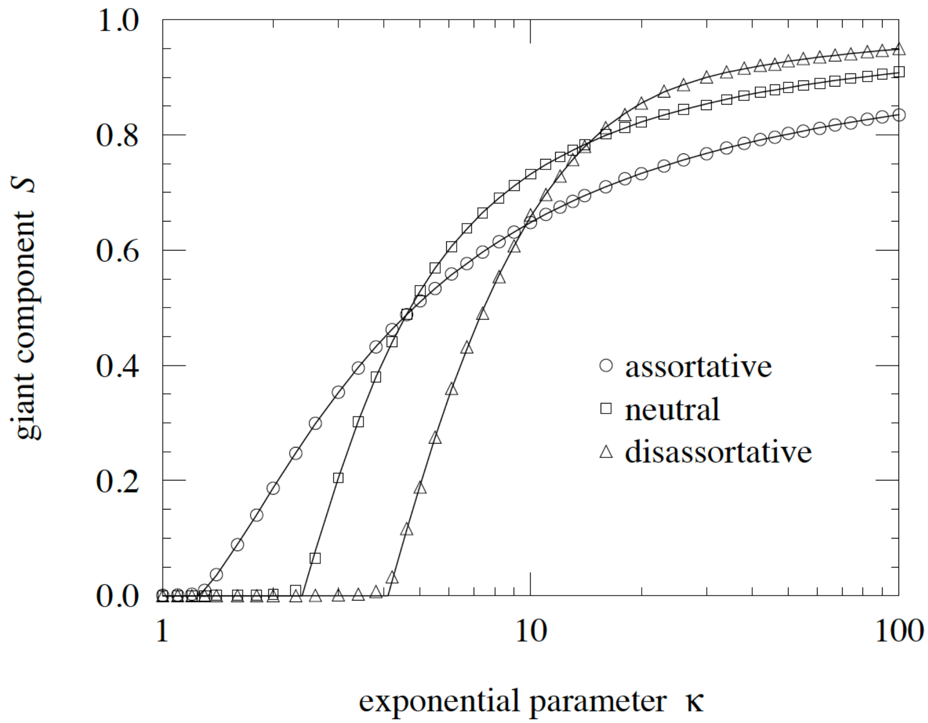
Possible Application of Normal Form: Finite Size Effects

Can normal form theory help us predict the finite size scaling for the network model?



Static networks with degree correlations

- Are degree correlations responsible for the infinite order phase transition?
- We can form static networks with degree correlations, they seem to have finite order transition (plots from Newman, *Phys Rev Lett. (2003)*)
- Assortative = Positive degree correlation



Other Kosterlitz-Thouless Transitions

Inverse Square Ising Model

$$\mathcal{H} = \sum_{I>j} \frac{J}{|i-j|^{\sigma+1}} S_i S_j, \quad \sigma = 1$$

Kondo Problem

$$\mathcal{H} = \sum_{\mathbf{k}\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - J\mathbf{S} \cdot \sum_{\mathbf{k}\mathbf{k}'\sigma\sigma'} c^\dagger_{\mathbf{k}\sigma} \sigma_{\sigma\sigma'} c_{\mathbf{k}'\sigma}$$

Inverse Square Ising Renormalization (Kosterlitz *Phys. Rev. Lett.* (1976))

$$\frac{d(p^2/T)}{dl} = -p^2/T[4(K\tau)^2 + \sigma - 1]$$
$$\frac{d(K\tau)}{dl} = -K\tau(p^2/T - 1)$$

p^2/T –"Topological Charge"

$K\tau$ – "Fugacity"

Normal form:

$$\frac{dt}{d\ell} = y^2(1+t)$$
$$\frac{dy}{d\ell} = ty$$

Different universality class than the XY model!!