Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
In Class Exercises
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1. **Hysteresis and Barkhausen Noise.** (Scaling) ⓘ

   Hysteresis is associated with abrupt phase transitions. Supercooling and superheating are examples (as temperature crosses T_c). Magnetic recording, the classic place where hysteresis is studied, is also governed by an abrupt phase transition – here the hysteresis in the magnetization, as the external field H is increased (to magnetize the system) and then decreased again to zero. Magnetic hysteresis is characterized by crackling (Barkhausen) electromagnetic noise. This noise is due to avalanches of spins flipping as the magnetic interfaces jerkily are pushed past defects by the external field (much like earthquake faults jerkily responding to the stresses from the tectonic plates). It is interesting that when dirt is added to this abrupt magnetic transition, it exhibits the power-law scaling characteristic of continuous transitions.

   Our model of magnetic hysteresis (unlike the experiments) has avalanches and scaling only at a special critical value of the disorder $R_c \sim 2.16$ (Figure 12.14). The integrated probability distribution $D(S, R)$ has a power law $D(S, R_c) \propto S^{-\bar{\tau}}$ at the critical point (where $\bar{\tau} = \tau + \sigma\beta\delta$ for our model) but away from the critical point takes the *scaling form*

   $$D(S, R) \propto S^{-\bar{\tau}} \mathcal{D}(S^{\sigma} (R - R_c)). \tag{1}$$

   Note from eqn (12.25) that at the critical disorder $R = R_c$ the distribution of avalanche sizes is a power law $D(S, R_c) = S^{-\bar{\tau}}$. The scaling form controls how this power law is altered as R moves away from the critical point. From Figure 12.14 we see that the main effect of moving above R_c is to cut off the largest avalanches at a typical largest size $S_{\max}(R)$, and another important effect is to form a 'bulge' of extra avalanches just below the cut–off.

   Using the scaling form from eqn 12.25, with what exponent does $S_{\max}$ diverge as $r = (R_c - R) \to 0$? (Hint: At what size S is $D(S, R)$, say, one millionth of $S^{-\bar{\tau}}$?) *Given $\bar{\tau} \approx 2.03$, how does the mean $\langle S \rangle$ and the mean-square $\langle S^2 \rangle$ avalanche size scale with $r = (R_c - R)$?* (Hint: Your integral for the moments should have a lower cutoff S_0, the smallest possible avalanche, but no upper cutoff, since that is provided by the scaling function $\mathcal{D}$. Assume $\mathcal{D}(0) > 0$. Change variables to $Y = S^{\sigma} r$. Which moments diverge?)

2. **Scaling and corrections to scaling.** (Condensed matter) ⓟ

Near critical points, the self-similarity under rescaling leads to characteristic power-law singularities. These dependences may be disguised, however, by less-singular corrections to scaling.

An experiment measures the susceptibility $\chi(T)$ in a magnet for temperatures T slightly above the ferromagnetic transition temperature T_c. They find their data is fit well by the form

$$\begin{aligned}\chi(T) = {} & A(T-T_c)^{-1.25} + B + C(T-T_c) \\ & + D(T-T_c)^{1.77}.\end{aligned} \tag{2}$$

(a) *Assuming this is the correct dependence near* T_c*, what is the critical exponent* γ*?*

When measuring functions of two variables near critical points, one finds universal scaling functions. The whole function is a prediction of the theory.

The pair correlation function $C(r,T) = \langle S(x)S(x+r)\rangle$ is measured in another, three-dimensional system just above T_c. It is found to be spherically symmetric, and of the form

$$C(r,T) = r^{-1.026}\mathcal{C}(r(T-T_c)^{0.65}), \tag{3}$$

where the function $\mathcal{C}(x)$ is found to be roughly $\exp(-x)$.

(b) *What is the critical exponent* ν*? The exponent* η*?*

3. **Superconductivity and the renormalization group.** (Condensed matter) ⓘ

Ordinary superconductivity happens at a rather low temperature; in contrast to phonon energies (hundreds of degrees Kelvin times k_B) or electronic energies (tens of thousands of degrees Kelvin), phonon-mediated superconductivity in most materials happens below a few Kelvin. This is largely explained by the BCS theory of superconductivity, which predicts that the transition temperature for weakly-coupled superconductors is

$$T_c = 1.764\,\hbar\omega_D \exp\left(-1/Vg(\varepsilon_F)\right), \tag{4}$$

where ω_D is a characteristic phonon frequency, V is an attraction between electron pairs mediated by the phonons, and $g(\varepsilon_F)$ is the density of states (DOS) of the electron gas (eqn **??**) at the Fermi energy. If V is small, $\exp\left(-1/Vg(\varepsilon_F)\right)$ can be exponentially small, explaining why materials often have to be so cold to go superconducting.

Superconductivity was discovered decades before it was explained. Many looked for explanations which would involve interactions with phonons, but there was a serious obstacle. People had studied the interactions of phonons with electrons, and had shown that the system stays metallic (no superconductivity) *to all orders in perturbation theory.*

(a) *Taylor expand T_c (eqn 12.58) about $V = 0^+$ (about infinitesimal positive V). Guess the value of all the terms in the Taylor series. Can we expect to explain superconductivity at positive temperatures by perturbing in powers of V?*

There are two messages here.

  - Proving something to all orders in perturbation theory does not make it true.

- Since phases are regions in which perturbation theory converges (see Section **??**), the theorem is not a surprise. It is a condition for a metallic phase with a Fermi surface to exist at all.

In recent times, people have developed a renormalization-group description of the Fermi liquid state and its instabilities[1] (see note **??** on p. **??**). Discussing Fermi liquid theory, the BCS theory of superconductivity, or this renormalization-group description would take us far into rather technical subjects. However, we can illustrate all three by analyzing a rather unusual renormalization-group flow.

Roughly speaking, the renormalization-group treatment of Fermi liquids says that the Fermi surface is a fixed-point of a coarse-graining in *energy*. That is, they start with a system space consisting of a partially-filled band of electrons with an energy width W, including all kinds of possible electron–electron repulsions and attractions. They coarse-grain by perturbatively eliminating (integrating out) the electronic states near the edges of the band,

$$W' = (1-\delta)W, \tag{5}$$

incorporating their interactions and effects into altered interaction strengths among the remaining electrons. These altered interactions give the renormalization-group flow in the system space. The equation for W gives the change under one iteration ($n = 1$); we can pretend n is a continuous variable and take $\delta n \to 0$, so $(W' - W)/\delta \to \mathrm{d}W/\mathrm{d}n$, and hence

$$\mathrm{d}W/\mathrm{d}n = -W. \tag{6}$$

When they do this calculation, they find the following.

  - The non-interacting Fermi gas we studied in Section **??** is a *fixed point of the renormalization group*. All interactions are zero at this fixed-point. Let V represent one of these interactions.[2]
  - The fixed-point is unstable to an attractive interaction $V > 0$, but is stable to a repulsive interaction $V < 0$.

- Attractive forces between electrons grow under coarse-graining and lead to new phases, but repulsive forces shrink under coarse-graining, leading back to the metallic free Fermi gas.

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[1] There are also other instabilities of Fermi liquids. Charge-density waves, for example, also have the characteristic $\exp(-1/aV)$ dependence on the coupling V.

[2] V will be the pairing between opposite-spin electrons near the Fermi surface for superconductors.

This is quite different from our renormalization-group treatment of phase transitions, where *relevant* directions like the temperature and field were unstable under coarse-graining, whether shifted up or down from the fixed-point, and other directions were *irrelevant* and stable (Fig. 12.8). For example, the temperature of our Fermi gas is a relevant variable, which rescales under coarse-graining like

$$\begin{aligned} T' &= (1 + a\delta)T, \\ \mathrm{d}T/\mathrm{d}n &= aT. \end{aligned} \tag{7}$$

Here $a > 0$, so the effective temperature becomes larger as the system is coarse-grained. How can they get a variable V which grows for $V > 0$ and shrinks for $V < 0$?

- When they do the coarse-graining, they find that the interaction V is *marginal*: to linear order it neither increases nor decreases. The next allowed term in the Taylor series near the fixed-point gives us the coarse-grained equation for the interaction:

$$\begin{aligned} V' &= (1 + b\delta V)V, \\ \mathrm{d}V/\mathrm{d}n &= bV^2. \end{aligned} \tag{8}$$

- They find $b > 0$.

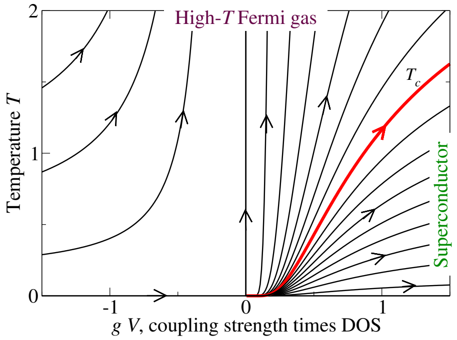

Fig. 1 Fermi liquid theory renormalization-group flows. The renormalization flows defined by eqns 12.61 and 12.62. The temperature T is relevant at the free Fermi gas fixed-point; the coupling V is marginal. The distinguished curve represents a phase transition boundary $T_c(V)$. Below T_c, for example, the system is superconducting; above T_c it is a (finite-temperature) metal.

(b) *True or false? (See Fig. 12.24.)*

(T) (F) For $V > 0$ (attractive interactions), the interactions get stronger with coarse-graining.

(T) (F) For $V < 0$ (repulsive interactions), coarse-graining leads us back to the free Fermi gas, explaining why the Fermi gas describes metals (Section **??***).*

(T) (F) Temperature is an irrelevant variable, but dangerous.

(T) (F) The scaling variable

$$x = TV^{1/\beta\delta} \tag{9}$$

is unchanged by the coarse-graining (second equations in 12.61 and 12.62), where β *and* δ *are universal critical exponents;*[3] *hence* x *labels the progress along the curves in Fig. 12.24 (increasing in the direction of the arrows).*

(T) (F) The scaling variable

$$y = T\exp\left(a/(bV)\right) \tag{10}$$

is unchanged by the coarse-graining, so each curve in Fig. 12.24 has a fixed value for y*.*

Now, without knowing anything about superconductivity, let us presume that our system goes superconducting at some temperature $T_c(V)$ when the interactions are attractive. When we coarse-grain a system that is at the superconducting transition temperature, we must get another system that is at its superconducting transition temperature.

(c) *What value for* a/b *must they calculate in order to get the BCS transition temperature (eqn 12.58) from this renormalization group? What is the value of the scaling variable (whichever you found in part (b)) along* $T_c(V)$*?*

Thus the form of the BCS transition temperature at small V, eqn 12.58, can be explained by studying the Fermi gas *without reference to the superconducting phase*!

4. **Singular corrections to scaling.** (Condensed matter) ③

The renormalization group says that the number of relevant directions at the fixed point in system space is the number of parameters we need to tune to see a critical point, and that the critical exponents depend on the eigenvalues of these relevant directions. Do the irrelevant directions matter?

Let the Ising model in zero field be described by flow equations

$$\mathrm{d}t_\ell/\mathrm{d}\ell = t_\ell/\nu, \qquad \mathrm{d}u_\ell/\mathrm{d}\ell = -y u_\ell \tag{11}$$

where t_ℓ describes the renormalization of the reduced temperature $t = (T_c - T)/T_c$ after a coarse-graining by a factor $b = \exp(\ell)$, and u and u_ℓ represent a slowly-decaying irrelevant perturbation under the renormalization group. In Fig. 12.8, one may view t as the expanding eigendirection running roughly horizontally, and u as the contracting, irrelevant coordinate running roughly vertically. Thus our model starts with a value u associated to the distance in system space between R_c and S^*.

(a) *What is the invariant combination* $z = ut^\omega$ *that stays constant under the renormalization group? What is* ω *in terms of the eigenvalues* $-y$ *and* $1/\nu$*?*

Properties near critical points have universal power law singularities, but the corrections to these power laws also have universal properties predicted by the renormalization

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[3] Note that here δ is not the infinitesimal change in parameter.

group. These come in two types – *analytic* corrections to scaling and *singular* corrections to scaling.

Let us consider corrections to the susceptibility. In analogy with other systems we have studied, we would expect that the susceptibility

$$\chi(t, u) = t^{-\gamma} X(z) \tag{12}$$

with $X(z)$ a universal function of the invariant combination you found in part (a). As a function of t, $\chi(t, u)$ has singularities at small t. But we expect properties to be analytic as we vary u, since the irrelevant direction is not being tuned to a special value, so we expect that a Taylor series of $\chi(t, u)$ in powers of u should make sense. Since $z \propto u$, we thus expect that $X(z)$ will be an analytic function of z for small z.[4]

(b) *Show that for small t, your z from part (a) goes to zero. Taylor expand $X(z)$. What corrections do you predict for the susceptibility from the first and second-order terms in the series?* These are the singular corrections to scaling due to the irrelevant perturbation u.

An Ising magnet on a sample holder is loaded into a magnetometer, and the susceptibility is measured[5] at zero external field as a function of reduced temperature $t = (T - T_c)/T_c$. It is found to be well approximated by

$$\begin{aligned}\chi(T) = At^{-1.24} + Bt^{-0.83} + Ct^{0.42} \\ + D + Et + \ldots\end{aligned} \tag{13}$$

You may ignore any errors due to the magnetometer.

(c) *The exponent $\omega \approx 0.407$ for the 3D Ising universality class, and $\gamma \approx 1.237$. Which terms are explained as singular corrections to scaling?*

(d) *Can you provide a physical interpretation for the terms in eqn 12.16 that are not explained by your theory? For example, how do we expect the susceptibility of the sample holder to depend on temperature?*

So far, we have relied on universality and rescaling to derive the universal power laws and scaling forms for properties near critical points. We can derive these in a mathematical way by including the flows of the predictions along with the flows of the control parameters under coarse-graining:

$$\begin{aligned}\mathrm{d}\chi_\ell/\mathrm{d}\ell &= -(\gamma/\nu)\chi_\ell, \\ \mathrm{d}t_\ell/\mathrm{d}\ell &= t_\ell/\nu, \\ \mathrm{d}u_\ell/\mathrm{d}\ell &= -y u_\ell\end{aligned} \tag{14}$$

How do we derive the universal scaling function $X(z)$ from these renormalization group flows? Consider the flows illustrated in Fig. 12.8, except now with a third dimension

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[4] Had we used a scaling variable $Z = tu^{1/\omega}$, for example, we would not have expected the corresponding scaling function to be analytic in small Z.

[5] The accuracy of the quoted exponents is not experimentally realistic.

involving the prediction χ. Consider a point (t_0, u_0, χ_0) in the system space, and the invariant curve defined by $z = u_0 t_0^\omega$ (dashed lines). Our renormalization group allows us to calculate $\chi_\ell(t_\ell, u_\ell)$ along these curves – relating the behavior everywhere near the critical manifold (vertical swath flowing toward S^*) to the properties along the outgoing trajectories, which approach closer and closer to the unstable manifold (the horizontal swath flowing away from S^*.

For example, we can define the universal scaling function $X(z)$ (for positive time t) to be the χ_{ℓ^*} where the flow crosses $t_{\ell^*} = 1$.

(e) *Solve eqns 12.17 for* u_ℓ *and* t_ℓ. *Setting* $t_{\ell^*} = 1$, *what is* u_{ℓ^*} *in terms of your invariant combination* z*?*

So we label each invariant scaling curve by the value of the vertical position u_{ℓ^*} where it crosses $t_{\ell^*} = 1$.

(f) *Solve eqns 12.17 for* $\chi_{\ell^*}(1, u_{\ell^*})$, *in terms of* z, t_0, *and* $\chi_0(t_0, u_0)$. *Use your solution to solve for the physical behavior* $\chi_0(t_0, u_0)$ *in terms of* t *and* $X(z)$. *Express* $X(z)$ *in terms of* $\chi_{\ell^*}(1, u_{\ell^*})$.

Remember the critical manifold is co-dimension one (or two, if you include temperature and external field), and the unstable manifold is dimension one (or two) – so we get universal predictions for a huge variety of systems, by observing the outgoing trajectories near a narrow tube or surface emitted from the fixed point.

5. **Bifurcation theory.** (Mathematics) ⓘ

Dynamical systems theory is the study of the time evolution given by systems of differential equations. Let $\mathbf{x}(t)$ be a vector of variables evolving in time t, let $\boldsymbol{\lambda}$ be a vector of parameters governing the differential equation, and let $\mathbf{F}_\lambda(\mathbf{x})$ be the differential equations

$$\dot{\mathbf{x}} \equiv \frac{\partial \mathbf{x}}{\partial t} = \mathbf{F}_\lambda(\mathbf{x}). \tag{15}$$

The typical focus of the theory is not to solve the differential equations for general initial conditions, but to study the qualitative behavior. In general, they focus on *bifurcations*—special values of the parameters $\boldsymbol{\lambda}$ where the behavior of the system changes qualitatively.

(a) *Consider the differential equation in one variable* $x(t)$ *with one parameter* μ*:*

$$\dot{x} = \mu x - x^3. \tag{16}$$

Show that there is a bifurcation at $\mu_c = 0$, *by showing that an initial condition with small, non-zero* $x(0)$ *will evolve qualitatively differently at late times for* $\mu > 0$ *versus for* $\mu < 0$. Hint: Although you can solve this differential equation explicitly, we recommend instead that you argue this qualitatively from the bifurcation diagram in Fig. 12.23; a few words should suffice.

Dynamical systems theory has much in common with equilibrium statistical mechanics of phases and phase transitions. The liquid–gas transition is characterized by external parameters $\boldsymbol{\lambda} = (P, T, N)$, and has a current state described by $\mathbf{x} = (V, E, \mu)$. Equilibrium phases correspond to fixed-points ($x^*(\mu)$ with $\dot{x}^* = 0$) in the dynamics, and phase transitions correspond to bifurcations.[6] For example, the power laws we find near continuous phase transitions have simpler analogues in the dynamical systems.

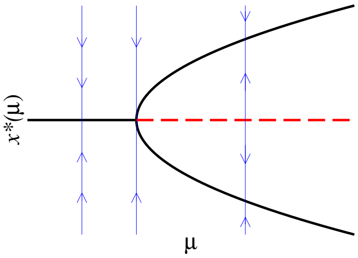

Fig. 2 Pitchfork bifurcation diagram. The flow diagram for the pitchfork bifurcation (eqn 12.48). The dashed line represents unstable fixed-points, and the solid thick lines represent stable fixed-points. The thin lines and arrows represent the dynamical evolution directions. It is called a pitchfork because of the three tines on the right emerging from the handle on the left.

(b) *Find the critical exponent* β *for the pitchfork bifurcation, defined by* $x^*(\mu) \propto (\mu - \mu_c)^\beta$ *as* $\mu \to \mu_c$.

Bifurcation theory also predicts universal behavior; all pitchfork bifurcations have the same scaling behavior near the transition.

(c) *At what value* λ_c *does the differential equation*

$$\dot{m} = \tanh(\lambda m) - m \tag{17}$$

have a bifurcation? Does the fixed-point value $m^*(\lambda)$ *behave as a power law* $m^* \sim |\lambda - \lambda_c|^\beta$ *near* λ_c *(up to corrections with higher powers of* $\lambda - \lambda_c$*)? Does the value of* β *agree with that of the pitchfork bifurcation in eqn 12.48?*

Just as there are different universality classes for continuous phase transitions with different renormalization-group fixed points, there are different classes of bifurcations each with its own *normal form.* Some of the other important normal forms include the saddle-node bifurcation,

$$\dot{x} = \mu - x^2, \tag{18}$$

transcritical exchange of stability,

$$\dot{x} = \mu x - x^2, \tag{19}$$

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[6] In Section **??**, we noted that inside a phase all properties are analytic in the parameters. Similarly, bifurcations are values of λ where non-analyticities in the long-time dynamics are observed.

and the Hopf bifurcation,

$$\begin{aligned} \dot{x} &= (\mu - (x^2 + y^2))x - y, \\ \dot{y} &= (\mu - (x^2 + y^2))y + x. \end{aligned} \tag{20}$$