Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
In Class Exercises
Last correction at September 7, 2017, 9:43 am

1. **Bifurcation theory.** (Mathematics) ⓘ

Dynamical systems theory is the study of the time evolution given by systems of differential equations. Let $\mathbf{x}(t)$ be a vector of variables evolving in time t, let $\boldsymbol{\lambda}$ be a vector of parameters governing the differential equation, and let $\mathbf{F}_\lambda(\mathbf{x})$ be the differential equations

$$\dot{\mathbf{x}} \equiv \frac{\partial \mathbf{x}}{\partial t} = \mathbf{F}_\lambda(\mathbf{x}). \tag{1}$$

The typical focus of the theory is not to solve the differential equations for general initial conditions, but to study the qualitative behavior. In general, they focus on *bifurcations*—special values of the parameters $\boldsymbol{\lambda}$ where the behavior of the system changes qualitatively.

(a) *Consider the differential equation in one variable $x(t)$ with one parameter μ:*

$$\dot{x} = \mu x - x^3. \tag{2}$$

Show that there is a bifurcation at $\mu_c = 0$, by showing that an initial condition with small, non-zero $x(0)$ will evolve qualitatively differently at late times for $\mu > 0$ versus for $\mu < 0$. Hint: Although you can solve this differential equation explicitly, we recommend instead that you argue this qualitatively from the bifurcation diagram in Fig. 1; a few words should suffice.

Dynamical systems theory has much in common with equilibrium statistical mechanics of phases and phase transitions. The liquid–gas transition is characterized by external parameters $\boldsymbol{\lambda} = (P, T, N)$, and has a current state described by $\mathbf{x} = (V, E, \mu)$. Equilibrium phases correspond to fixed-points ($x^*(\mu)$ with $\dot{x}^* = 0$) in the dynamics, and phase transitions correspond to bifurcations.[1] For example, the power laws we find near continuous phase transitions have simpler analogues in the dynamical systems.

[1] In Section 8.3, we noted that inside a phase all properties are analytic in the parameters. Similarly, bifurcations are values of λ where non-analyticities in the long-time dynamics are observed.

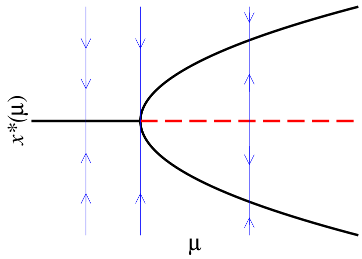

Fig. 1 Pitchfork bifurcation diagram. The flow diagram for the pitchfork bifurcation (eqn 2). The dashed line represents unstable fixed-points, and the solid thick lines represent stable fixed-points. The thin lines and arrows represent the dynamical evolution directions. It is called a pitchfork because of the three tines on the right emerging from the handle on the left.

(b) *Find the critical exponent β for the pitchfork bifurcation, defined by $x^*(\mu) \propto (\mu - \mu_c)^\beta$ as $\mu \to \mu_c$.*

Bifurcation theory also predicts universal behavior; all pitchfork bifurcations have the same scaling behavior near the transition.

(c) *At what value λ_c does the differential equation*

$$\dot{m} = \tanh(\lambda m) - m \tag{3}$$

have a bifurcation? Does the fixed-point value $m^*(\lambda)$ behave as a power law $m^* \sim |\lambda - \lambda_c|^\beta$ near λ_c (up to corrections with higher powers of $\lambda - \lambda_c$)? Does the value of β agree with that of the pitchfork bifurcation in eqn 2?

Just as there are different universality classes for continuous phase transitions with different renormalization-group fixed points, there are different classes of bifurcations each with its own *normal form.* Some of the other important normal forms include the saddle-node bifurcation,

$$\dot{x} = \mu - x^2, \tag{4}$$

transcritical exchange of stability,

$$\dot{x} = \mu x - x^2, \tag{5}$$

and the Hopf bifurcation,

$$\begin{aligned} \dot{x} &= (\mu - (x^2 + y^2))x - y, \\ \dot{y} &= (\mu - (x^2 + y^2))y + x. \end{aligned} \tag{6}$$

2. **Singular corrections to scaling.** (Condensed matter) ③

The renormalization group says that the number of relevant directions at the fixed point in system space is the number of parameters we need to tune to see a critical point, and that the critical exponents depend on the eigenvalues of these relevant directions. Do the irrelevant directions matter?

Let the Ising model in zero field be described by flow equations

$$\mathrm{d}t_\ell/\mathrm{d}\ell = t_\ell/\nu, \qquad \mathrm{d}u_\ell/\mathrm{d}\ell = -y u_\ell \tag{7}$$

where t_ℓ describes the renormalization of the reduced temperature $t = (T_c - T)/T_c$ after a coarse-graining by a factor $b = \exp(\ell)$, and u and u_ℓ represent a slowly-decaying irrelevant perturbation under the renormalization group. In Fig. 12.8, one may view t as the expanding eigendirection running roughly horizontally, and u as the contracting, irrelevant coordinate running roughly vertically. Thus our model starts with a value u associated to the distance in system space between R_c and S^*.

(a) *What is the invariant combination $z = ut^\omega$ that stays constant under the renormalization group? What is ω in terms of the eigenvalues $-y$ and $1/\nu$?*

Properties near critical points have universal power law singularities, but the corrections to these power laws also have universal properties predicted by the renormalization group. These come in two types – *analytic* corrections to scaling and *singular* corrections to scaling.

Let us consider corrections to the susceptibility. In analogy with other systems we have studied, we would expect that the susceptibility

$$\chi(t, u) = t^{-\gamma} X(z) \tag{8}$$

with $X(z)$ a universal function of the invariant combination you found in part (a). (We shall derive this scaling form in Exercise 3.) As a function of t, $\chi(t, u)$ has singularities at small t. But we expect properties to be analytic as we vary u, since the irrelevant direction is not being tuned to a special value, so we expect that a Taylor series of $\chi(t, u)$ in powers of u should make sense. Since $z \propto u$, we thus expect that $X(z)$ will be an analytic function of z for small z.[2]

(b) *Show that for small t, your z from part (a) goes to zero. Taylor expand $X(z)$. What corrections do you predict for the susceptibility from the first and second-order terms in the series?* These are the singular corrections to scaling due to the irrelevant perturbation u.

An Ising magnet on a sample holder is loaded into a magnetometer, and the susceptibility is measured[3] at zero external field as a function of reduced temperature

[2] Had we used a scaling variable $Z = t u^{1/\omega}$, for example, we would not have expected the corresponding scaling function to be analytic in small Z.

[3] The accuracy of the quoted exponents is not experimentally realistic.

$t = (T - T_c)/T_c$. It is found to be well approximated by

$$\chi(T) = At^{-1.24} + Bt^{-0.83} + Ct^{-0.42} + D + Et + \dots \tag{9}$$

You may ignore any errors due to the magnetometer.

(c) *The exponent* $\omega \approx 0.407$ *for the 3D Ising universality class, and* $\gamma \approx 1.237$. *Which terms are explained as singular corrections to scaling?*

(d) *Can you provide a physical interpretation for the terms in eqn 9 that are not explained by singular corrections to scaling? For example, how do we expect the susceptibility of the sample holder to depend on temperature?* These are examples of *analytic* corrections to scaling.

3. **Singular corrections to scaling and the renormalization group.** (Condensed matter) ③

In this exercise, we derive the form of the scaling function (eqn 8) for the effects of *irrelevant* operators on the properties of systems near critical points (see Exercise 2). Remember that irrelevant directions shrink under coarse-graining. Let χ be the susceptibility of the Ising model, as a function of the reduced temperature $t = T_c - T$ and some irrelevant operator u:

$$\begin{aligned} \mathrm{d}\chi_\ell/\mathrm{d}\ell &= -(\gamma/\nu)\chi_\ell, \\ \mathrm{d}t_\ell/\mathrm{d}\ell &= t_\ell/\nu, \\ \mathrm{d}u_\ell/\mathrm{d}\ell &= -y u_\ell \end{aligned} \tag{10}$$

How do we derive the universal scaling function $X(z)$ from these renormalization group flows? Consider the flows illustrated in Fig. 12.8, except now with a third dimension involving the prediction χ. Consider a point (t_0, u_0, χ_0) in the system space, and the invariant curve defined by $z = u_0 t_0^\omega$ (dashed lines). Our renormalization group allows us to calculate $\chi_\ell(t_\ell, u_\ell)$ along these curves – relating the behavior everywhere near the critical manifold (vertical swath flowing toward S^*) to the properties along the outgoing trajectories, which approach closer and closer to the unstable manifold (the horizontal swath flowing away from S^*.

For example, we can define the universal scaling function $X(z)$ (for positive time t) to be the χ_{ℓ^*} where the flow crosses $t_{\ell^*} = 1$.

(a) *Solve eqns 10 for* u_ℓ *and* t_ℓ. *Setting* $t_{\ell^*} = 1$, *what is* u_{ℓ^*} *in terms of your invariant combination* z*?*

So we label each invariant scaling curve by the value of the vertical position u_{ℓ^*} where it crosses $t_{\ell^*} = 1$.

(b) *Solve eqns 10 for* $\chi_{\ell^*}(1, u_{\ell^*})$, *in terms of* z, t_0, *and* $\chi_0(t_0, u_0)$. *Use your solution to solve for the physical behavior* $\chi_0(t_0, u_0)$ *in terms of* t *and* $X(z)$. *Express* $X(z)$ *in terms of* $\chi_{\ell^*}(1, u_{\ell^*})$. Does your answer agree with the form in eqn 8?

Remember the critical manifold is co-dimension one (or two, if you include temperature and external field), and the unstable manifold is dimension one (or two) – so we get universal predictions for a huge variety of systems, by observing the outgoing trajectories near a narrow tube or surface emitted from the fixed point.