Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
Material for Week 5
Exercises due Tuesday Sep 26
Last correction at September 25, 2017, 4:10 pm

Pre-class Preparation

Thursday

Skim: Cardy, chapter 3, **The renormalization-group idea**, paying particular attention to section 3.2, *One-dimensional Ising model.*

Complex RG eigenvalues. *Cardy mentions that strange things would happen if any of the RG eigenvalues were complex. Argue that any such eigenvalues must be in complex conjugate pairs. How would the power law scaling we observe near critical points change if eigenvalues were complex?* See also the bold claims of Didier Sornette in Why stock markets crash: Critical events in complex financial systems and Complex Critical Exponents from Renormalization Group Theory of Earthquakes: Implications for Earthquake Predictions, Didier Sornette and Charles G. Sammis, J. Phys. I France **5**, 607-619 (1995). (Submit electronically by 9:30 Wednesday evening.)

Tuesday

Skim: Cardy, chapter 5, **The perturbative renormalization group**.

Operator product expansions. Wilson perturbs around the Gaussian fixed point, where we can calculate everything. Many treatments plunge directly into calculations, calculating the new terms in the free energy by integrating over shells in momentum space. Notice that Cardy instead sets up the technology of the operator product expansion – an N-point correlation function, when two positions approach one another, can be written as a sum over $N-1$-point correlation functions times a particular, known function involving the universal operator product expansion coefficients c_{ijk}. Find the equation giving the RG equations for the Landau theory coefficients in terms of the c_{ijk}. Find the section where Cardy explains how to calculate c_{ijk} for the $4-\epsilon$ expansion. Do the operator product expansion coefficients depend on the universality class? (Submit electronically by 9:30 Monday evening.)

Exercises

1. **Eigenvectors near the RG Fixed Point.** ③

   The critical exponents in the renormalization group are given by the eigenvalues of the RG transformation linearized near the fixed point. What do the eigenvectors mean?

Consider a two-dimensional Ising model with two parameters, a nearest-neighbor bond[1] $K = J/T$ and a next-neighbor interaction $K_2 = J_2/T$ lying along the diagonal bonds.

$$\mathcal{H} = -K \sum_{i,j} S_{i,j}S_{i+1,j} + S_{i,j}S_{i,j+1} - K_2 \sum_{i,j} S_{i,j}S_{i+1,j+1} + S_{i,j}S_{i+1,j-1} \tag{1}$$

If we decimate to the 'black' squares of a checkerboard (say, $i+j$ even), we get a new square-lattice Hamiltonian rotated by 45° coarse-grained by a factor $b = \sqrt{2}$. The next-neighbor bond basically becomes a nearest-neighbor bond – it mostly renormalizes to zero in one step, and contributes its value to the new nearest-neighbor coupling. The deviation of the nearest-neighbor bond from the critical point K^*, we may crudely assume, rescales by a factor $b^{1/\nu}$ under coarse-graining (remember $K \sim J/T$) and then is increased by K_2. So under one coarse-graining step

$$\begin{aligned} K' - K^* &= b^{1/\nu}(K - K^*) + K_2, \\ K_2' &= 0. \end{aligned} \tag{2}$$

(a) *Our crude renormalization-group flow is already linear. What is the fixed point? What is the Jacobian J about the fixed point? What are the eigenvalues λ_0 and λ_1?* (Let λ_1 be the relevant eigenvalue, greater than one.)

Our Jacobian matrix is not symmetric (or Hermitian), so it has two sets of eigenvectors – left eigenvectors $\hat{\boldsymbol{\ell}}_\alpha J = \lambda_\alpha \hat{\boldsymbol{\ell}}_\alpha$, and right eigenvectors $J\hat{\mathbf{r}}_\alpha = \lambda_\alpha \hat{\mathbf{r}}_\alpha$.

(b) *What are the left and right eigenvectors? Are the left eigenvectors orthonormal? Are they normal to the right eigenvector that has a different eigenvalue?*

(c) *Draw the flow in the (K, K_2) plane near the fixed point. Indicate the directions of the left eigenvectors and right eigenvectors in different colors. Also draw the boundary between the ferromagnetic and paramagnetic phase. How is this boundary related to the stable manifold of the fixed point? Is it related to any of the eigenvectors?*

Consider a new set of *scaling variables* u_α, given by the dot products of the displacement from the fixed point with the left eigenvectors:

$$u_\alpha = \boldsymbol{\ell}_\alpha \cdot (K - K^*, K_2) \tag{3}$$

(d) *Show that the phase boundary has $u_1 = 0$ (using the convention that λ_1 is the relevant direction). How do the coordinates u_α flow under the renormalization group?*

In general, there is a nonlinear transformation between the parameters $T, H, J_2, \ldots$ in a Hamiltonian and the natural coordinates $t(T,H,J_2), h(T,H,J_2), u(T,H,J_2)$ which flow simply under the renormalization group. This coordinate change is one of the contributors to analytic corrections to scaling.

[1] Instead of thinking of a Hamiltonian space with temperature as an extra parameter, it is convenient to work at fixed temperature, and mimic raising temperature by lowering the overall scale of the energy.

(e) *Are* u_0 *and* u_1 *relevant, irrelevant, or marginal? Which coordinate,* u_0 *or* u_1*, is the scaling variable corresponding to the reduced temperature* $t(K, K_2)$*? If we write a property of our system* $X(K, K_2) = X(K(u_0, u_1), K_2(u_0, u_1)) = u_1^x \mathcal{X}(u_0/u_1^y)$*, can there be any dependence on* u_0*, within our crude model? How does* X *vary near the phase boundary?*