Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
Material for Week 8
Exercises due Tuesday Oct. 24
Last correction at October 18, 2017, 3:00 pm

Pre-class Preparation

Thursday

Read Rayleigh-Bénard experiment in liquid helium; frequency locking and the onset of turbulence, J. Maurer and A. Libchaber, *Journal de Physique Lettres* **40**, L-419 (1979), and Read pp. 693–694, and appendix A1: The Universal Metric Properties of Nonlinear Transformations, Mitchell J. Feigenbaum, *J. Stat. Phys.* **21**, 669 (1979).

Polish. Feigenbaum's paper is clearly a work in progress. He had announced the key idea in an earlier paper, and appears pressured to write up current work because everyone else was getting excited. His numerical method, presented on the pages above, turns out to be much better than methods others tried later (and I tried while making up the exercise). *Browse the rest of the paper. There is a conjecture on page 687, that suggests that all of the eigenvalues of the linearization of the RG transformation, except δ, are expressable as powers of α. Is anything like this true of statistical mechanics models, to your knowledge? Do you believe him?* (Submit electronically by 9:30 Wednesday evening.)

Tuesday

Read: Crackling Crossover, James P. Sethna, *Nature Physics* **3**, 518-519 (2007).

Crossover scaling and high T_c. One of the most interesting theories of high-temperature superconductors is that they are due to an underlying quantum critical point. A remarkable number of them share a phase diagram where the superconducting 'dome' separates a Fermi liquid phase and an antiferromagnetic phase, with weird behavior seen in a region much like that shown in Cardy's figure 4.5. This weird behavior in the 'classical' region is usually called the 'pseudogap', although it is also characterized by several other properties.
Find a picture of this phase diagram, and a discussion about the relation with quantum criticality. Give the reference; was it accessible?
(Submit electronically by 9:30 Monday evening.)

Exercises

1. **The onset of chaos: Lowest order renormalization-group for period doubling.** ③

   In this exercise, we set up a low-order approximate renormalization group to study the period-doubling route to chaos of Fig. 12.17. Our goal is to estimate the scaling factors $\alpha \sim -2.5029$ and $\delta \sim 4.6692$ governing the self-similarity in space x and control parameter μ near the onset of chaos μ_∞, as discussed in Exercise 12.6. We shall extend the renormalization group we develop here to higher order in Exercise 2.

The period-doubling route to chaos is understood by a renormalization group that (as usual) has a coarse-graining step and a rescaling. The coarse-graining step 'decimates' the time series $\{x, g(x), g(g(x)), \dots\}$ by dropping every other point, replacing the function $g(x)$ by $g(g(x))$. The rescaling expands x about its maximum by a factor α. If we choose the maximum value of $g(x)$ to be at zero, the renormalization group sends g to the function $T[g]$, where

$$T[g](x) = \alpha g\left(g(x/\alpha)\right). \tag{1}$$

(We explore the scaling and universality implied by this renormalization group in Exercise 12.22; our eqn 1 is the same as eqn 12.69) in that exercise, except there our functions were symmetric about $x = \nicefrac{1}{2}$.)

Our functions $g(x)$ are 'one-humped maps', with a parabolic maximum at zero. To lowest order, let us approximate $g(x)$ by a parabola centered at the origin

$$g(x) \approx G_0 + G_1 x^2. \tag{2}$$

where $G_1 < 0$.

We shall approximate the fixed point g^* of our renormalization group by demanding that $T[g^*](x) = g^*(x)$ at two points, $x = 0$ and $x = 1$. We have three constants, α, G_0^*, and G_1^*, so for convenience we set G_0^* to one.

(a) *Use the fixed-point condition at* $x = 0$ *to show that* $\alpha = 1/g^*(1) = 1/(1 + G_1^*)$.

(b) *Use the fixed-point condition at* $x = 1$ *to give an equation for* G_1^*, *substituting in your equation for* α *above.* (Hint: The equation simplifies to a sixth-order polynomial.)

We expect $\alpha \approx -2.5$, so since our approximate $\alpha = 1/(1 + G_1^*)$, we expect $G_1^* \approx 1/\alpha - 1 \approx -1.4$.

(c) *Plot the quantity from part (b) that must be zero: does it have a root near* $G_1 = -1.4$? *Numerically solve your equation from part (b) for the root closest to* -1.4. *What is* α *in our approximation?* (Your approximation for α should be within a few percent of the correct value $\alpha = -2.5029\dots$; your value for G_1^* should be within a few percent of -1.4 and not too far from the true value of the quadratic term at the fixed point, $-1.5276\dots$.)

In statistical mechanics, we find the universal critical exponents by linearizing the renormalization-group flows about the fixed point and finding directions that grow. Here the exponent $\delta = 4.669\dots$ describes the fastest growing direction in function space: $T[g^* + \epsilon\psi](x) - g^*(x) = \delta\psi(x)$. That is, we add a perturbation $g(x) = g^*(x) + \epsilon\psi(x)$ and study to linear order in ϵ how the perturbation grows under T. The lowest-order perturbation to our parabola adds an overall constant $G_0 \to 1 + \epsilon$. Our function $g^*(x)$ is fixed at both $x = 0$ and $x = 1$. Let us check the growth of our perturbation at $x = 0$, which should grow by a factor of approximately δ when our renormalization-group transformation is applied.

(d) *Using* $g(x) = (1+\epsilon) + G_1 x^2$, *write the formula for the term in* $T[g](0)$ *linear in* ϵ *as a function of* G_1 *and* α. *Insert your fixed-point values for* G_1 *and* α *from part (c). What is your estimate for* δ*?* (Your approximation for δ should be within a few percent of the correct value $\delta = 4.669\ldots$.)

2. **The onset of chaos: Full renormalization-group calculation.**[1] ③

In this exercise, we implement Feigenbaum's numerical scheme [1, pp. 693-694] for finding high-precision values of the universal constants

$$\begin{aligned} \alpha &= -2.502907875095892822283902873218 \\ \delta &= 4.669201609102990671853203820466, \end{aligned} \tag{3}$$

that quantify the scaling properties of the period-doubling route to chaos (Fig. 12.17, Exercise 12.6). This extends the lowest-order calculation of Exercise 1.

Our renormalization group operation (Exercises 12.22 and 1) coarse-grains in time taking $g \to g \circ g$, and then rescales distance x by a factor of α. Centering our functions at $x = 0$, this leads to $T[g](x) = \alpha g\left(g(x/\alpha)\right)$ (eqn 1).

We shall solve for the properties at the onset of chaos by analyzing our function-space renormalization-group by expanding our functions in a power series

$$g(x) \approx 1 + \sum_{n=1}^{N} G_n x^{2n}. \tag{4}$$

Notice that we only keep even powers of x; the fixed point is known to be symmetric about the maximum, and the unstable mode responsible for the exponent δ will also be symmetric.

First, we must approximate the fixed point $g^*(x)$ and the corresponding value of the universal constant α. At order N, we must solve for α and the N polynomial coefficients G_n^*. We can use the $N+1$ equations fixing the function at equally spaced points in the positive unit interval:

$$T[g^*](x_m) = g^*(x_m), \qquad x_m = m/N,\ m = \{0, \ldots, N\}. \tag{5}$$

We can use the first of these equations to solve for α.

(a) *Show that the equation for* $m = 0$ *sets* $\alpha = 1/g^*(1)$.

We can use a root-finding routine to solve for G_n^*. (b) *Implement the other* N *constraint equations of eqn 5 in a form appropriate for your method of finding roots of nonlinear equations, substituting your value for* α *from part (a). Check that your routine at* $N = 1$

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[1] Hints for the computations can be found at the book Web site [2].

gives values for $\alpha \approx -2.5$ *and* $G_1^* \approx -1.5$. (These should reproduce the values from Exercise 1(c).)

(c) *Use a root-finding routine to calculate* α *for* $N = 1, \ldots, 9$. *Start the search at* $G_1^* = -1.5$, $G_n^* = 0$ $(n > 1)$ *to avoid landing at the wrong fixed point.* (If it is convenient for you to use high-precision arithmetic, continue to higher N.) *To how many decimal places can you reproduce the correct value for* α *in eqn 3?*

Now we need to solve for the renormalization group flows $T[g]$, linearized about the fixed point $g(x) = g^*(x) + \epsilon\psi(x)$. Feigenbaum tells us that $T[g^* + \epsilon\psi] = T[g^*] + \epsilon\mathcal{L}[\psi]$, where $\mathcal{L}$ is the linear operator taking $\psi(x)$ to

$$\mathcal{L}[\psi](x) = \alpha\psi(g^*(x/\alpha)) + \alpha g^{*\prime}(g(x/\alpha))\psi(x/\alpha). \tag{6}$$

(d) *Derive eqn 6.*

We want to find eigenfunctions that satisfy $\mathcal{L}[\psi] = \lambda\psi$. Again, we can expand $\psi(x)$ in a polynomial

$$\psi(x) = \sum_{n=0}^{N-1} \psi_n x^{2n} \qquad (\psi_0 \equiv 1). \tag{7}$$

We then approximate the action of $\mathcal{L}$ on ψ by its action at N points $\widetilde{x}_i$, that need not be the same as the N points x_m we used to find g^*. We shall use $\widetilde{x}_i = (i-1)/(N-1)$, $i = 1, \ldots, N$. (For $N = 1$, we use $\widetilde{x}_1 = 0$.) This leads us to a linear system of N equations for the coefficients ψ_n, using eqns 7 and 8:

$$: \sum_{n=0}^{N-1} \left[\alpha g(\widetilde{x}_i/\alpha)^{2n} + \alpha g'(g(\widetilde{x}_i/\alpha))(\widetilde{x}_i/\alpha)^{2n}\right] \psi_n = \lambda \sum_{n=0}^{N-1} \widetilde{x}_i^{2n}\psi_n \tag{8}$$

These equations for the coefficients ψ_n of the eigenfunctions of $\mathcal{L}$ is in the form of a *generalized eigenvalue problem*

$$\sum_n L_{in}\psi_n = \lambda \sum_n X_{in}\psi_n. \tag{9}$$

The solution to the generalized eigenvalue problem can be found from the eigenvalues of $X^{-1}L$, but most eigenvalue routines provide a more efficient and accurate option for directly solving the generalized equation given L and X.

(e) *Write a routine that calculates the matrices* L *and* X *implicitly defined by eqns 9 and 8. For* $N = 1$ *you should generate* 1×1 *matrices. For* $N = 1$, *what is your prediction for* δ*?* (These should reproduce the values from Exercise 1(d).)

(f) *Solve the generalized eigenvalue problem for* L *and* X *for* $N = 1, \ldots, 9$. *To how many decimal places can you reproduce the correct value for* δ *in eqn 3?*

References

[1] Feigenbaum, Mitchell (1979, 12). The universal metric properties of nonlinear transformations. *Journal of Statistical Physics*, **21**, 669–706.

[2] Sethna, J. P. and Myers, C. R. (2004). *Entropy, Order Parameters, and Complexity* computer exercises: Hints and software. http://www.physics.cornell.edu/sethna/StatMech/ComputerExercises.html.