Physics 7653: Statistical Physics
http://www.physics.cornell.edu/sethna/teaching/653/
Material for Week 11
Exercises due Tuesday Nov. 14
Last correction at November 14, 2017, 7:29 pm

Pre-class Preparation

Thursday

Read: Ginzburg-Landau Phenomenology, Ben Simons and Michal Kwasigroch, Chapter 2, Sections 2.1–2.3 and 2.5. (If you like, do the pre-class question first.)

(Mijo Ghosh).

1. **Vortices in the XY model.**[1] ⓟ

The Hamiltonian for the 2-dimensional XY model is the familiar

$$-\beta H = K \sum_{\langle ij \rangle} \mathbf{S}_i . \mathbf{S}_j$$

Here $\mathbf{S}_i = (\cos\theta_i, \sin\theta_i) \; \forall \; i$,where θ_i is the angle (a continuous variable) that the spin at location i makes with the horizontal. This model is also known as the rotor model. It is not difficult to see that:

$$-\beta H = K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

For low temperatures, we can think of θ as a field and write the Hamiltonian as:

$$-\beta H = \frac{K}{2} \int \mathrm{d}^2\mathbf{x} \;\; (\nabla\theta)^2$$

The gradient expansion describes the energy cost of small deformations around the ground state and applies to configurations that can be continuously deformed to the uniformly ordered state. Berezinskii, and later Kosterlitz and Thouless, suggested that disordering can also be caused by "topological" defects that cannot be obtained from such deformations.

Since the angle describing the orientation of a spin is defined up to an integer multiple of 2π, it is possible to construct spin configurations in which the traversal of a closed path will see the angle rotate by $2\pi n$. The integer n is the **topological charge** enclosed by the path. The discrete nature of the charge makes it impossible to find a continuous

[1] Problem developed by Mitrajyoti Ghosh, 2017.

deformation which returns the state to the uniformly ordered configuration in which the charge is zero.

The elementary defect, or **vortex**, has a unit charge. In completing a circle centered on the defect the orientation of the spin changes by $\pm 2\pi$. If the radius r of the circle is sufficiently large, the variations in angle will be small and the lattice structure can be ignored. By symmetry $\nabla\theta$ has uniform magnitude and points along the azimuthal direction.

Let us look at one of these vortices at the origin, and assume that the others are far away - and define a vector field $\mathbf{v} = \nabla\theta$ around it.

a *Why is* $\nabla \times \mathbf{v} \neq 0$ *despite the fact that* $\mathbf{v}$ *is the gradient of a scalar?* Hint: What is the value of θ at the vortex? *Convince yourself that*

$$\nabla \times \mathbf{v} = 2\pi \hat{\mathbf{k}}\, \delta^2(\mathbf{x})$$

(Here $\hat{\mathbf{k}}$ is the unit vector perpendicular to the plane of the spins.)

(b) *By varying the hamiltonian with respect to θ, find the equation of motion for θ. What does this tell you about* $\nabla .\mathbf{v}$ *?*

(c) *Define a new vector* $\mathbf{E}$ *such that* $E_x = v_y$ *and* $E_y = -v_x$*. Find* $\nabla \times \mathbf{E}$ *and* $\nabla.\mathbf{E}$ *.* (You can look up the expressions for curl in 2D, if you aren't familiar with it. *Are the expressions familiar? Is it clear now why n is called a topological* ***charge****?*

(Submit electronically to Mijo Ghosh (mg2338@cornell.edu) by 9:30 Wednesday evening; cc Sethna.)

Tuesday

Read: TASI Lectures on the Conformal Bootstrap, David Simmons-Duffin, especially chapter 8 (intro to OPE) and sections 9.1-10.5,
and Bootstrapping Mixed Correlators in the 3D Ising Model, Filip Kos, David Pol,and David Simmons-Duffin, *Journal of High Energy Physics* **2014**, 109 (2014).

(Peter Cha). (Peter Cha (pjc277@cornell.edu).)

Exercises

- **Long-range Ising.** (Dimension dependence) ③

  The one-dimensional Ising model can have a finite-temperature transition if we give each spin an interaction with distant spins.

  Long-range forces in the 1d Ising model. *Consider an Ising model in one dimension, with long-range ferromagnetic bonds*

$$\mathcal{H} = \sum_{i>j} \frac{J}{|i-j|^\sigma} S_i S_j. \tag{1}$$

For what values of σ will a domain wall between up and down spins have finite energy? Suggest a bound for the 'lower critical power law' for this long-range one-dimensional Ising model, below which a ferromagnetic state is only possible when the temperature is zero. (Hint: Approximate the sum by a double integral. Avoid $i = j$.)

The long-range 1D Ising model at the lower critical power law has a transition that is closely related to the Kosterlitz-Thouless transition. It is in the same universality class as the famous (but obscure) Kondo problem in quantum phase transitions. And it is less complicated to think about and less complicated to calculate with than either of these other two cases.

- Cardy, exercise 4.4 (Crossover in an Ising slab.)