

Physics 7653: Statistical Physics
<http://www.physics.cornell.edu/sethna/teaching/653/>
Material for Week 13
Exercises due Thursday Nov. 30
Last correction at November 27, 2017, 7:57 am
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Pre-class Preparation

Thursday

Have a great Thanksgiving!

Tuesday

No class Tuesday Nov. 28.

Thursday

Last day of class – homework only.

Exercises

• **Is the fixed point unique? Period doubling.** ③

Is the fixed-point of the renormalization group unique? (It seems unlikely that coarse-graining the Ising model in momentum space gives the same fixed-point Hamiltonian as real-space decimation on a square lattice. One would be spherically symmetric, the other has a square symmetry. Naturally, both look the same on long length and time scales, but their short-distance behavior is different.) If not, and there are many alternative fixed-points in system space describing a phase transition, can any system at the critical point be a fixed point, for a suitable renormalization group?

We shall answer this question for the particular case of the period-doubling onset of chaos. In particular, we shall investigate what happens to the renormalization-group fixed point as we change coordinates. There is no reason to expect that Nature measures distances x in the same way, though, as we do. We could equally well decimate and rescale in a different coordinate $y = \phi(x)$, where we assume ϕ is smooth, monotone increasing, has a smooth inverse ϕ^{-1} . We will be considering one-humped maps with a maximum at $x = 0$, so we shall assume $\phi(0) = 0$ to keep the maximum at zero.

(a) Give the formula for the function $\tilde{g}(y)$ corresponding to the map $g(x)$, in terms of g and ϕ . Hint: You need to find the x -value from y , then apply g to get the new x , and then find the new y . Show that the inverse formula is for g in terms of ϕ and \tilde{g} is

$$g(x) = \phi^{-1}\tilde{g}(\phi(x)). \quad (1)$$

If $g(x)$ is at the onset of chaos (showing universal scaling behavior), will $\tilde{g}(y)$ also be at the onset?

Recall from Exercises 12.22 and ?? that the transition to chaotic motion (Fig. 12.17) of one-humped maps $g(x)$ is understood using a renormalization group that decimates in time by a factor of two using $g(g(x))$, and rescaling the coordinate x by a factor of $\alpha \approx -2.5$:

$$T[g](x) = \alpha g(g(\alpha^{-1}x)). \quad (2)$$

Let g^* be the fixed point $T[g^*] = g^*$ of our renormalization group in the space of one-humped maps.

(b) Write a renormalization group transformation \tilde{T} for which \tilde{g}^* is a fixed point. (Hint: Use the formula $T[g^*] = g^*$, and the formula for g^* in terms of \tilde{g}^* and ϕ from part (a). Change variables to y and solve for $\tilde{g}(y)$ on the right-hand side.) Show that \tilde{T} can be viewed as a decimation plus a nonlinear ‘stretching’ function $\tilde{\alpha}(y)$, $\tilde{T} = \tilde{\alpha} \circ \tilde{g} \circ \tilde{g} \circ \tilde{\alpha}^{-1}(y)$. What is $\tilde{\alpha}(y)$?

So there is an infinite family of plausible, nonlinear renormalization-group transformations, with an infinite family of fixed points given taking our original fixed point $g^*(x)$ and changing variables $x \rightarrow \phi(x)$.

Can we change variables so that our fixed point is a parabola (as in the logistic map)? Can we make any one-humped map at the onset of chaos a fixed point, by choosing an appropriate change of coordinates?

There is an elegant proof that this is not possible. Our one-humped map has a fixed point $g^*(x^*) = x^*$, and also many periodic orbits¹ $g^{*[2^n]}(x_n) = g^*(g^*(\dots(x_n)\dots)) = x_n$. with period $2, 4, 8, \dots$ (Beware! g^* is a fixed point in function space under the transformation T . x^* is a fixed point on the real line under the function g .)

(c) Show that $dx/dy = \phi^{-1}'(\phi(x)) = 1/(dy/dx) = 1/\phi'(x)$. If x^* is a fixed point of g^* , what is the fixed point y^* of \tilde{g}^* ? Find the slope $d\tilde{g}^*/dy|_{y^*}$ at the new fixed point, and show that it equals the slope $g^{*'}(x^*)$ at the old fixed point. Similarly, show that $dg^{*[2^n]}/dx|_{x_n} = d\tilde{g}^{*[2^n]}/dy|_{y_n}$. (Note that $dg(g(\dots(x)\dots))/dx = g'(g(\dots))g'(\dots)\dots$, so the product of the derivatives along any period orbit is invariant under coordinate changes.)

So, unless the logistic map happens to have the same derivative as g^* at their respective fixed points, there can be no coordinate transform taking one to the other, and there is no reason to think there is some nonlinear renormalization-group transformation that has the logistic map as a fixed point. (Later we shall find a more direct way to see that the logistic map has different behavior than any fixed point.) Indeed, all the fixed points of all the iterates of a map would have to agree with the iterates of g^* to allow for such a transformation. So our critical surface in function space, where one-humped maps

¹Remember the period doubling bifurcations, which start with a stable 2^n cycle and end with a stable 2^{n+1} cycle surrounding an unstable 2^n cycle, forming 2^n little pitchforks. All of these unstable 2^n cycles survive to the onset of chaos.

are poised at the onset of chaos, has two subsurfaces – the maps that can be formed under coordinate changes (and could be fixed points), and the maps that cannot be fixed points.

We can gain more insight into these two types of critical points by considering the renormalization-group flow of functions \tilde{g}^* . (It is a fixed point of \tilde{T} in part (b), but it must flow to g^* under our original renormalization-group transformation T .) Consider maps near to g^* , formed by picking coordinate transformations ϕ that are near to the identity:

$$\phi(x) = x + \epsilon \sum_{p=1}^{\infty} \phi_p x^p. \quad (3)$$

(d) Write $\phi^{-1}(y)$ to linear order in ϵ , as a similar sum. Calculate the change $\tilde{g}^*(x) - g^*(x)$ to linear order in ϵ . (Hint: Start with the equation $\phi(g^*(x)) = \tilde{g}^*(\phi(x))$.) Express it as a sum $\epsilon \sum_{p=1}^{\infty} \phi_p \Psi_p(x)$.

(e) Show that $g^*(g^*(x/\alpha)) = g^*(x)/\alpha$. By differentiating $T[g^*][x]$, show that $g^{*'}(x) = g^{*'}(g(x/\alpha))g^{*'}(x/\alpha)$. Calculate the change $T[\tilde{g}^*](x) - g^*(x)$ to linear order in ϵ . Use your first two formulas to show that the change can be written $\epsilon \sum_{p=1}^{\infty} \phi_p \alpha^{1-p} \Psi_p(x)$.

You have shown that $\Psi_p(x)$ is an eigenfunction of T , with eigenvalue α^{1-p} . You have *also* shown that only these eigenvalues and eigenfunctions can be generated by infinitesimal changes of coordinates. Feigenbaum conjectured in his early paper [1, p.687] that these eigenfunctions were the only ones. Our argument in part (c) suggests that there must be functions near to g^* that cannot be reached by a change of coordinates.

The lowest few eigenvalues λ_n of T , calculated for perturbations in the even subspace, are approximately

$$\begin{aligned} \{\lambda_1, \lambda_2, \dots\} = \{ & 4.67, 1.00, 0.160, -0.124, -0.0573, 0.0255, -0.0101, 0.00407, 0.00240, \\ & -0.00163, 0.000649, -0.000259, 0.000104, -0.0000735, \\ & -0.0000413, 0.0000165 \dots \}. \end{aligned} \quad (4)$$

For your convenience, we provide the powers of $\alpha \approx -2.503$, so

$$\begin{aligned} \alpha^{1-p} = \{ & 1.00, -0.400, 0.160, -0.0638, 0.0255, -0.0102, 0.00407, \\ & -0.00163, 0.000649, -0.000259, 0.000104, -0.0000414, 0.0000165, \dots \}. \end{aligned} \quad (5)$$

(f) Is λ_4 a power of α ? Was Feigenbaum correct? Give the next few eigenvalues λ_n that are not given by coordinate changes. Which powers p are missing from the list? Do those powers correspond to even, or odd perturbations to g^* ?

So, at least at the period-doubling onset of chaos, there are two kinds of critical points, those that can be reached by a smooth change of coordinates (and could be fixed points of a different renormalization group), and those that cannot. We propose what may be a new nomenclature. We shall call the first *redundant* critical points and the second

(more common) category *singular* critical points. More generally, there are three sources of corrections to scaling. Smooth coordinate transformations in the control variables (here the parameter μ in the logistic map) generate what we call *analytic corrections to scaling*; these can give corrections with integer power-laws (like the analytic background in the specific heat), or corrections that are combinations of integers and the main critical exponents. There are *singular corrections* to scaling, which are generated by irrelevant eigenvectors of the renormalization group that are new, irrational numbers not related to the other exponents (perhaps like λ_4 in part (f)). Finally, there are *redundant* corrections to scaling, which correspond to systems that could have been fixed points of a different renormalization group. The redundant corrections to scaling will involve combinations of powers of relevant critical exponents.

How in practice can we tell the special ‘redundant’ critical points from the ‘singular’ ones?

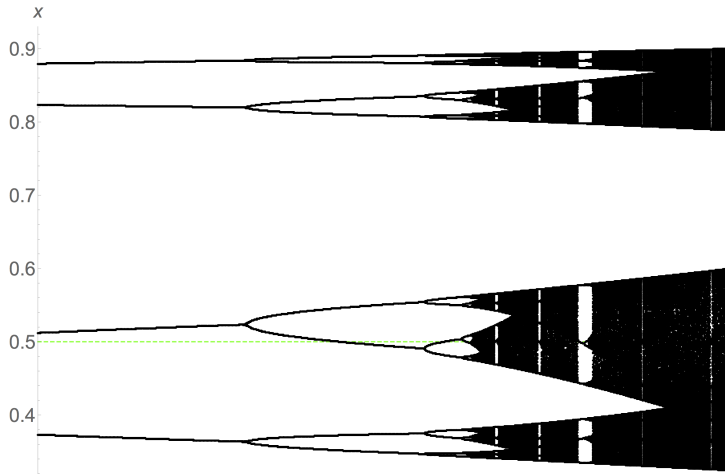


Fig. 1 Expanded bifurcation diagram. The attractor at the fixed point g^* has a perfect self-similarity upon flipping and rescaling about the function maximum $x = 0.5$ (green-dashed line). The lower first-tier ‘branch’ of the bifurcation diagram maps onto the entire diagram when rescaled vertically by α and horizontally by δ ; the same happens to the lower branch (whose upper second-tier branch maps onto the entire lower first-tier branch), and so on. Any other map will have this property only asymptotically, near the maximum (green dashed line) and near the critical point.

Suppose we study a map at its critical point, which deviates from g^* primarily along the eigendirection corresponding to $\lambda_4 = -0.124$. This deviation will lead to changes in the spatial patterns seen in the attractor (Fig. 1); for example, the ratio α_n of the widths of the n^{th} and $(n + 1)^{\text{st}}$ tier branch will not be precisely α .

Let the $\psi_n(x)$ be the n^{th} eigenfunction of T .²

²We already have a name for the special eigenfunctions, so for example the eigenfunction associated with $\lambda_2 = 1 = \alpha^{1-p}$ with $p = 1$ is $\psi_2(x) = \Psi_1(x) = \psi_2(x)$.

(g) Consider an infinitesimal perturbation of the fixed-point $g^* + \epsilon\psi_4(x)$ along this direction. How does the leading correction to $\alpha_n - \alpha$ scale with λ_n ? (Hint: How does α_n change under one application of the renormalization group operator T ?)

Thus the special, redundant critical points have corrections to scaling that are expressible solely in terms of the relevant critical exponents.

Conjectures:

1. The allowed renormalization-group fixed point functions are precisely those accessible by analytic coordinate transformations $y = \phi(x)$, which are precisely those generated by moving along the scaling variables u_n^0 associated with the eigenvalues λ_n not given by powers of α (in a way similar to how infinitesimal Lie algebra symmetries are related to Lie group operations). In particular, a large change of variables will have corrections to scaling at the critical point only involving powers of α .
2. We conjecture that all corrections to scaling for one-humped maps [3] will be given by changing to ‘normal form’ coordinates. So, for the logistic map $f(x) = \mu x(1 - x)$, the traditional analysis suggests there are nonlinear scaling coordinates that satisfy $T(u_n) = \lambda u_n$; the curve in function space traced by the logistic map then has coordinates $u_\mu(\mu) = \mu - \mu_\infty + b_1(\mu - \mu_\infty)^2 + \dots$, $u_2(\mu) = a_2 + b_2(\mu - \mu_\infty) + c_2 \dots$, \dots . We are led to conjecture that a further change of coordinates $x \rightarrow \phi(x)$ allows us to remove all corrections involving the special eigendirections with $\lambda = \alpha^{1-p}$ for integer $p > 1$. That is, we conjecture that not only is there a redundant submanifold in the surface of critical points, but that the critical points can be ‘foliated’ into surfaces where the singular corrections are the same.
3. We conjecture that the separation between redundant and singular fixed points we define here have analogies in the Ising model and other critical points. (Indeed, we took the name from that literature [?], where it involves changes in the coordinates describing the fields.) See in particular reference [2], which answers the analogous question as to whether there is a renormalization group that can make any Ising critical point into its fixed point.
4. We note that, in the case of thermodynamic critical points, the distinction between control variables (like $t = T - T_c$ and $h = H/T$) and ‘results’ (like magnetization $M(t, h)$, entropy $S(t, h)$, and energy $E(t, h)$) are rather artificial; under a Legendre transformation we can take a system described by the Gibbs free energy $G(T, H)$ to a system described by the microcanonical entropy $S(E, H)$ or the Helmholtz free energy $A(T, M)$. We conjecture that Legendre transformations will convert the analytic corrections to redundant corrections in our nomenclature. If so, it may be better to treat these on an equal footing, and term them both ‘analytic’ corrections to scaling.
5. We conjecture that a subset of the irrelevant eigenvalues for other critical points will similarly be given in terms of combinations of relevant eigenvalues. We conjecture that analytic changes in the results variables can remove these corrections

to scaling. (Again, Nature does not tell you how you should measure your magnetization, just as it did not tell you how to measure your applied field.)

6. Many of the ideas here were prompted by an analysis of the onset of chaos from quasiperiodic motion [4, 5]. There the coordinate transformation taking the map to a simple rotation was numerically straightforward, central to understanding maps below the onset of chaos, and led to self-similar, nonanalytic maps at the onset of chaos. The coordinate transformations that connected two points on the critical manifold were not analytic, but numerically had a continuous first derivative. We conjecture that the corrections to scaling at the quasiperiodic onset of chaos will again separate into redundant and singular components.

References

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