Functional Renormalization Group

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Statistical Physics 2

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Contents

- Partition function (finite temperature): $Z = \mathrm{Tr} e^{-\beta H} = \oint \mathcal{D}\varphi \, e^{-S(\varphi)}$.
  - Expectation values: $\langle (\star) \rangle = \frac{1}{Z} \mathrm{Tr} \, (\star) \, e^{-\beta H} = \frac{1}{Z} \oint \mathcal{D}\varphi \, (\star) \, e^{-S(\varphi)}$.

Introduction

Functional Methods and Notation

- Partition function (finite temperature): $Z = \mathrm{Tr} e^{-\beta H} = \oint \mathcal{D}\varphi \, e^{-S(\varphi)}$.
  - Expectation values: $\langle (\star) \rangle = \frac{1}{Z} \mathrm{Tr}\,(\star)\, e^{-\beta H} = \frac{1}{Z} \oint \mathcal{D}\varphi\,(\star)\, e^{-S(\varphi)}$.
- Generating functional (zero temperature): $\mathcal{Z}[J] = \int \mathcal{D}\varphi \, e^{-S[\varphi] + \int d^d x \varphi(x) J(x)}$.
  - Vacuum correlation functions: $\langle \varphi(x_1) ... \varphi(x_n) \rangle = \frac{1}{\mathcal{Z}[0]} \left. \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} \equiv \mathcal{Z}^{(n)}[0]$.

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Statistical Physics	**Quantum Field Theory**
Partition Function: $Z = \mathrm{Tr}\, e^{-\beta H}$	Generating Functional: $\mathcal{Z}[J] = \int \mathcal{D}\varphi \, e^{-S[\varphi] + (J,\varphi)}$
Helmholtz Free Energy: $F = -\frac{1}{\beta} \ln Z$	Schwinger Functional: $\mathcal{W}[J] = \ln \mathcal{Z}[J]$
Gibbs Free Energy: $G = F + pV$	Effective Action: $\Gamma[\phi] = \sup_J \left(\int J\phi - \mathcal{W}[J] \right)$

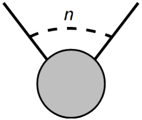
Introduction

Functional Methods and Notation

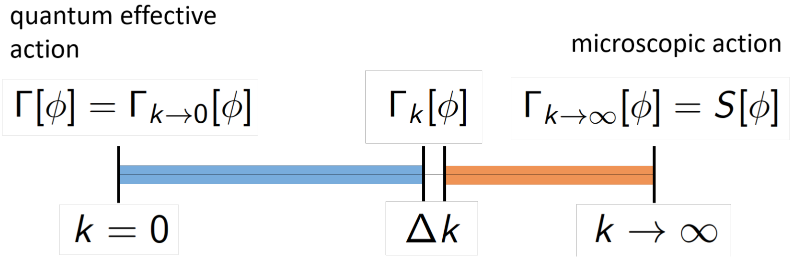
Properties of the effective action:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - \mathcal{W}[J] \right)$$

- If maximum exists: $\phi = \langle \varphi \rangle_J$.
- 1PI diagrams: $\Gamma^{(n)}[\phi]$.
  - Full propagator: $G(x,z) = \left(\Gamma^{(2)}[\phi]\right)^{-1}(x,y)$.

$n > 2$: $\Gamma^{(n)}[\phi](x_1 \ldots x_n) =$ 

Idea:


quantum effective action
$\Gamma[\phi] = \Gamma_{k\to 0}[\phi]$
$\Gamma_k[\phi]$
microscopic action
$\Gamma_{k\to\infty}[\phi] = S[\phi]$
$k = 0$
Δk
$k \to \infty$

- Regulated Schwinger Functional:

$$e^{\mathcal{W}_k[J]} \equiv \mathcal{Z}_k[J] = e^{-\Delta S_k[\frac{\delta}{\delta J}]}\,\mathcal{Z}[J] = \int_\Lambda \mathcal{D}\varphi\, e^{-S[\varphi]-\Delta S_k[\varphi]+\int d^d x J(x)\varphi(x)}$$

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$$\Gamma_k[\phi] = \sup_J \left(\int J\phi - \mathcal{W}_k[J] \right) - \Delta S_k[\phi]$$

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- Regulator Function:

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi(-q) R_k(q^2) \varphi(q)$$

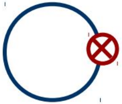
  - Conditions for $R_k(q^2)$: $\lim_{q^2/k^2 \to 0} R_k(q^2) > 0$, $\lim_{k^2/q^2 \to 0} R_k(q^2) = 0$, $\lim_{k \to \Lambda \to \infty} R_k(q^2) \to \infty$

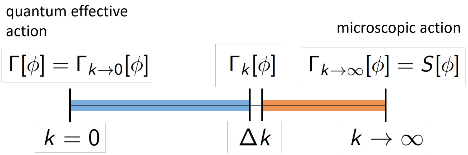
Wetterich equation

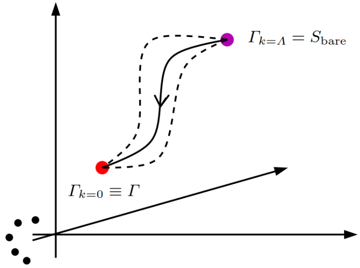
[Wetterich '93, *Phys. Lett.*, B301:90-94]

$$k\partial_k \Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\frac{k\partial_k R_k}{\Gamma_k^{(2)}[\phi] + R_k}$$

$$k\partial_k \Gamma_k = \frac{1}{2}$$







[Gies '06, hep-ph/0611146]

Flow Equation for the Effective Action

Truncation Schemes

- Derivative expansion:

$$\Gamma_k[\phi] = \int d^d x \left\{ V_k(\phi) + \frac{1}{2} Z_k(\rho) \partial_\mu \phi^a \partial^\mu \phi_a + \frac{1}{4} Y_k(\rho) \partial_\mu \rho \partial^\mu \rho + \mathcal{O}(\partial^4) \right\}$$

- Expansion in powers of the fields:

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \left(\prod_{j=0}^{n} d^d x_j \, [\phi(x_j) - \phi_0] \right) \Gamma_k^{(n)}(x_1, \ldots, x_n)$$

Challenge: Find optimized regulator [Litim 2000, hep-th/0005245].

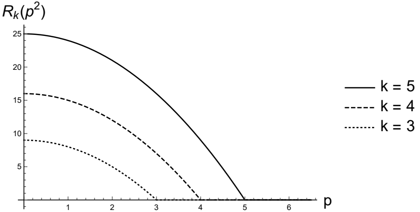
Ansatz:

- Local potential approximation of effective action:

$$\Gamma_k[\phi] = \int d^d x \left[V_k(\phi) + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \right]$$

- Flat cutoff regulator:

$$R_k(p^2) = (k^2 - p^2)\Theta(k^2 - p^2)$$
$$k\partial_k R_k(p^2) = 2k^2\Theta(k^2 - p^2)$$



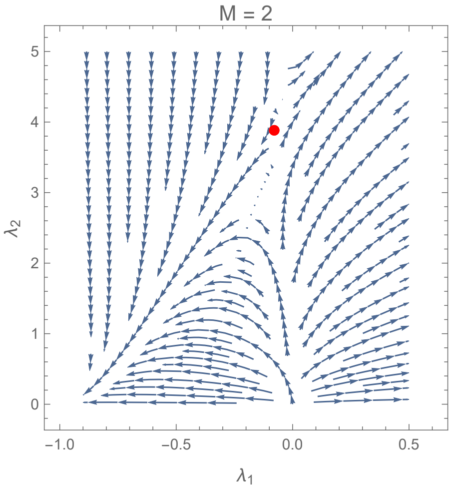
$O(1)$-Model

Expansion Ansatz and Flow Equation for the Effective Potential

- First order derivative expansion of effective action $\Rightarrow$ $\dot{\Gamma}[\phi]\Big|_{\phi=\text{const.}} = \dot{V}(\phi)\,\mathrm{Vol}_{\mathbb{R}^d}$
- Wetterich equation $\Rightarrow$ $\dot{V} = \frac{1}{\mathrm{Vol}_{\mathbb{R}^d}}\frac{1}{2}\mathrm{Tr}\,\dot{R}G$ with $G = \left(\Gamma^{(2)} + R\right)^{-1}$
- $\Gamma^{(2)}[\phi](p,q) = \left(V^{(2)}(\phi) + p^2\right)(2\pi)^d\,\delta^{(d)}(p+q)$
- $O(1)$-symmetric potential: $V = V(\rho)\,\text{with}\,\rho = \frac{\phi^2}{2}$

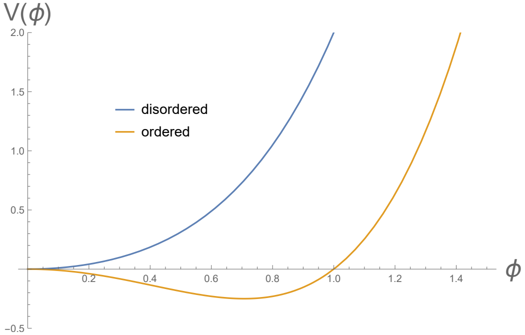
Flow equation for the effective potential

$$\dot{V}_k(\phi) = \frac{\Omega_d}{(2\pi)^d}\frac{k^{d+2}}{d}\frac{1}{V_k'(\rho)+2\rho V_k''(\rho)+k^2}$$



Flow from UV to IR.

- Effective potential: $V(\rho) = \sum_{i=0}^{M} \frac{\lambda_i}{i!} \rho^i$
- Dimensions: $D = 3$



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Order M	RG eigenvalue $-y_t$ of the critical direction	critical exponent $\nu = \frac{1}{y_t}$
2	-1.84256	0.542722
3	-1.68572	0.593218
4	-1.58551	0.630711

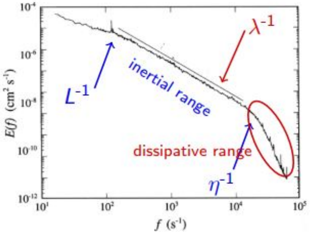
Expected value (7 loops, conformal mapping): [Guida, Zinn-Justin '98, arXiv:cond-mat/9803240]

$$\nu = 0.6304 \pm 0.0013$$

Turbulence:

- Kinetic energy spectrum:

$$E(k) \propto \frac{\epsilon^{2/3}}{k^{5/3}} \exp\left[-\mu(\lambda k)^{2/3}\right]$$



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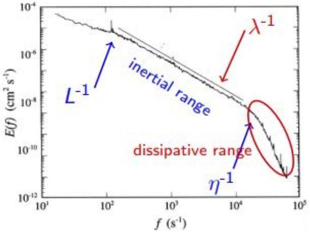

Asymptotic safety in Quantum Gravity:

- Einstein-Hilbert Truncation: $\Gamma_k = \frac{1}{16\pi G_k} \int d^4x \sqrt{g}\,[R + 2\Lambda_k] +$ gauge fixing terms
- Dimensionless couplings:
  - Newton Coupling: $g_k = k^2 G_k$
  - Cosmological Constant: $\lambda_k = k^{-2}\Lambda_k$

$\Rightarrow$ Non-Gaussian UV fixed point, attractive in g and λ!

[FRG Meeting, March 2017, Heidelberg. http://www.thphys.uni-heidelberg.de/~frg-meeting/]