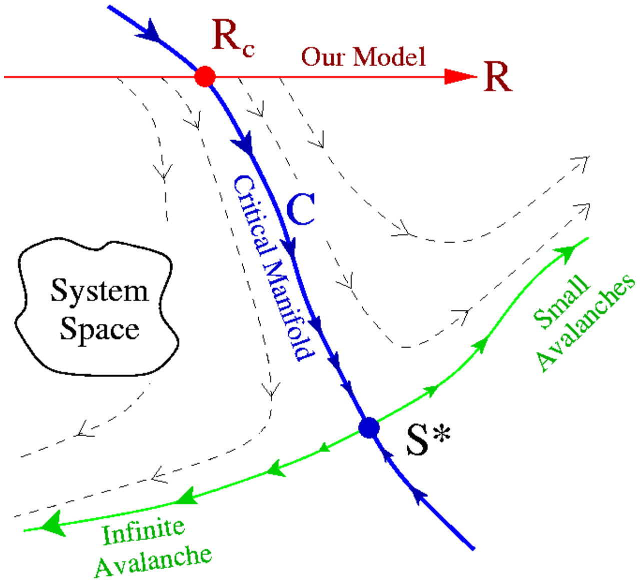
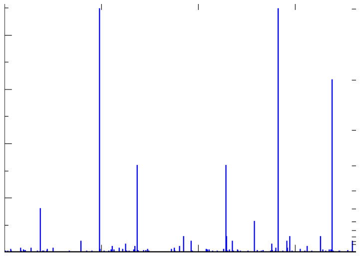
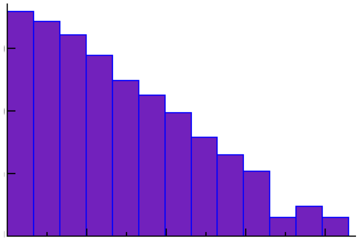
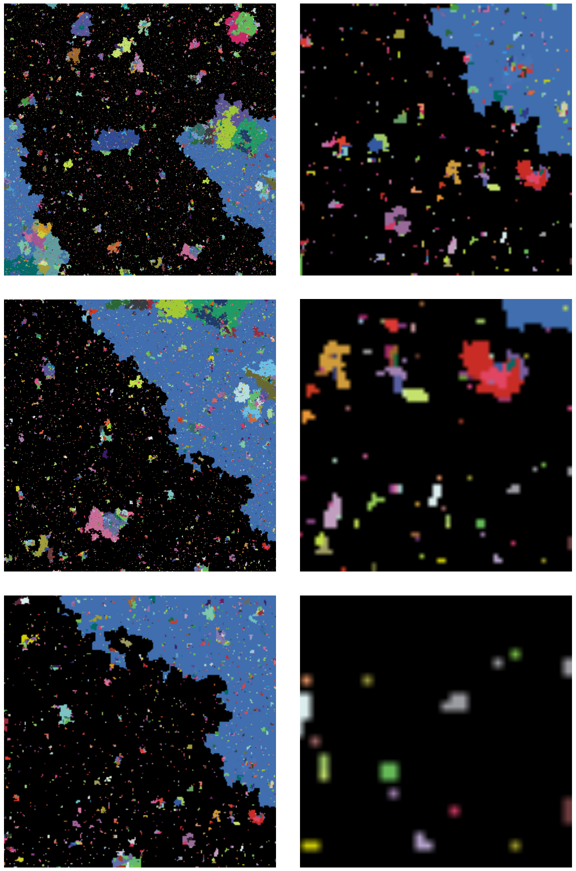
Continuous Phase Transitions

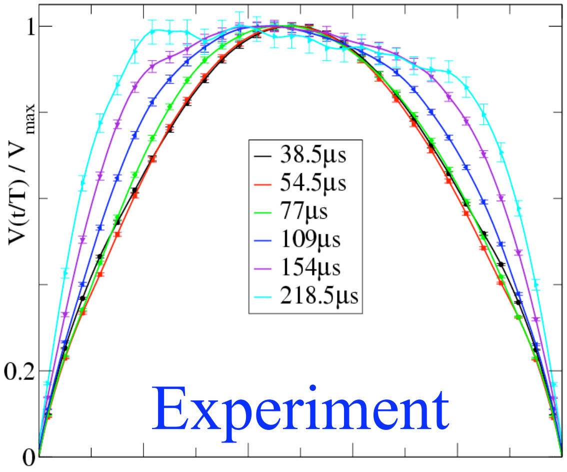
Jim Sethna, Physics 653, Fall 2010

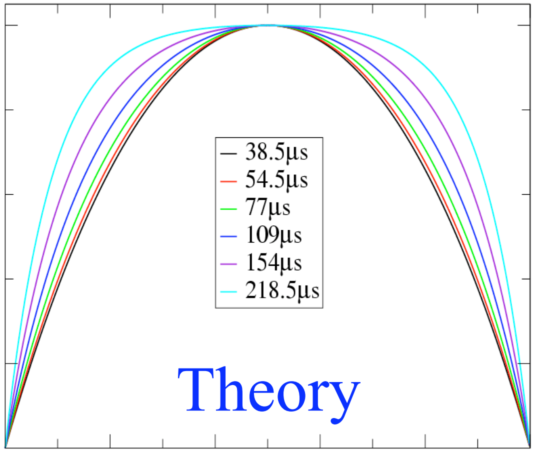








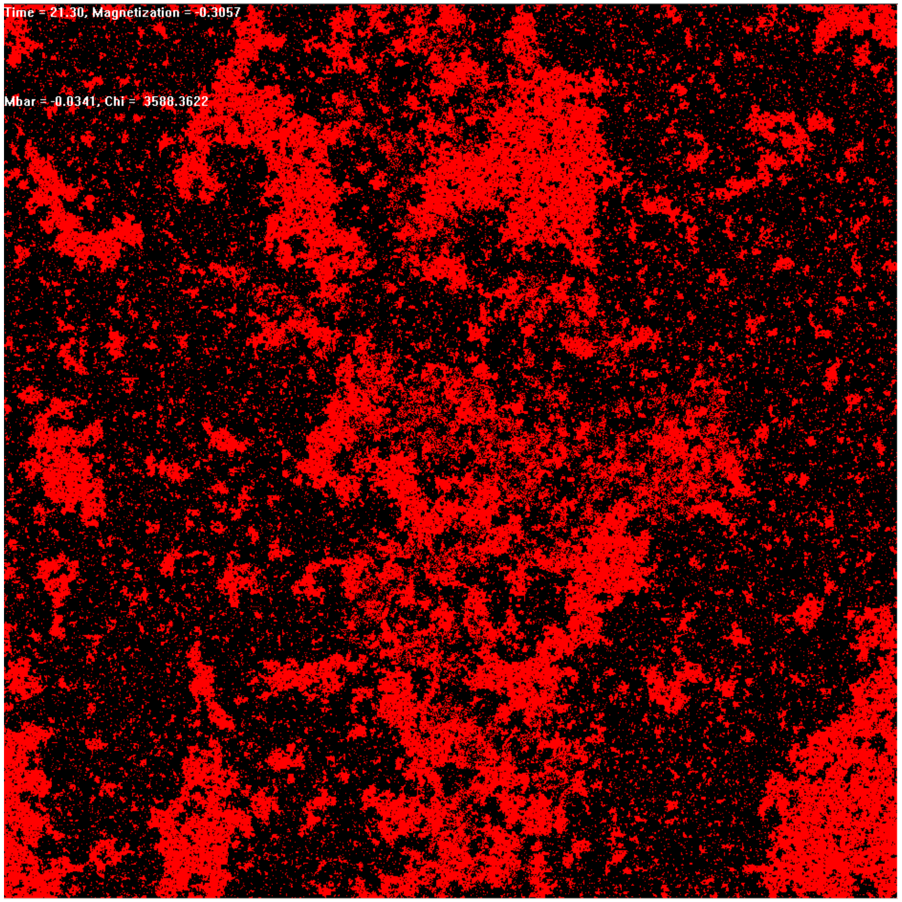




Yanjiun Chen, Stefanos Papanikolaou, Karin Dahmen, Olga Perković, Chris Myers, Matt Kuntz, Gianfranco Durin, Stefano Zapperi, …

The Ising Model at T_c

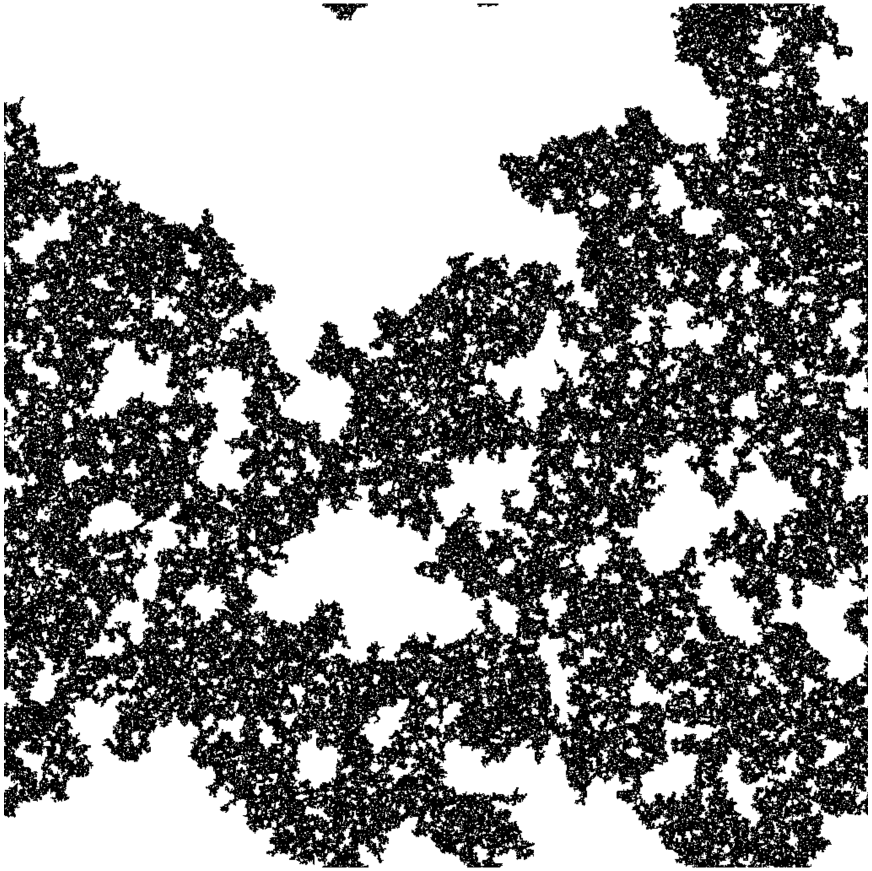
Structure on All Scales



- Continuous Phase Transition
- Competition Entropy vs. Energy
- Thermal Disorder
- High Temperature:
  Random
- Low Temperature
  Long-Range Order
- Critical Point
  - $T_c = 2/\log(1+\sqrt{2}) \sim 2.27$
  - Fluctuations on All Scales

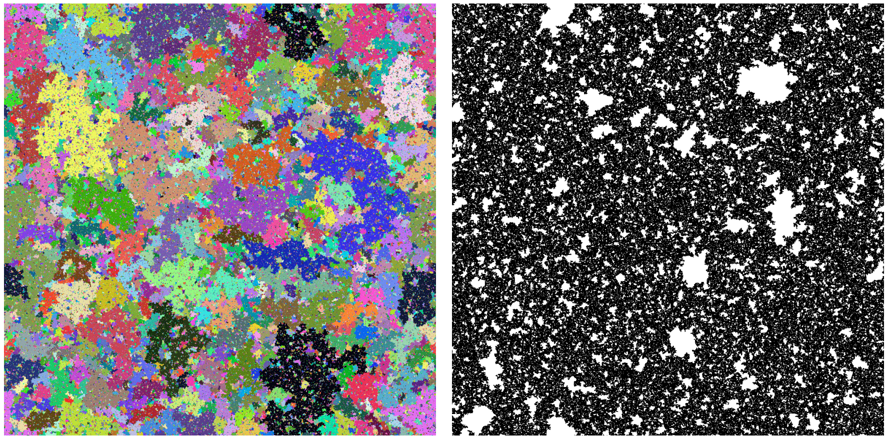
Percolation

Structure on All Scales



Largest Connected Cluster
$P=P_c$

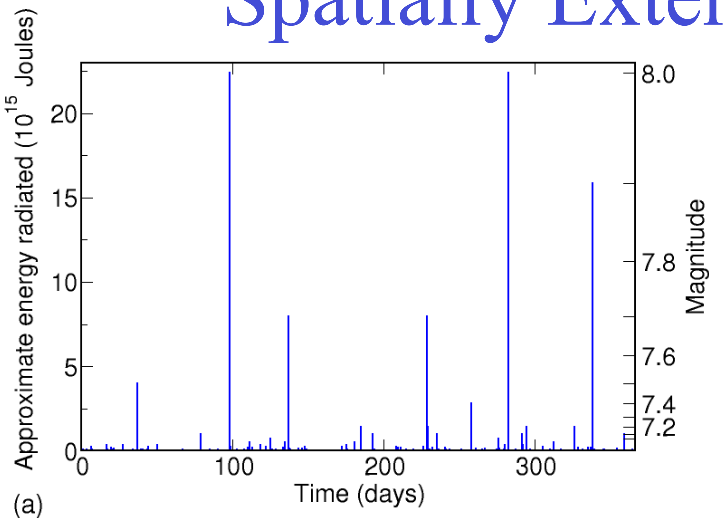
- Connectivity Transition
- Punch Holes at Random,
  Probability 1-P
  P_c =1/2 Falls Apart
  (2D, Square Lattice, Bond)
- Static (Quenched) Disorder



P=0.49 P=0.51

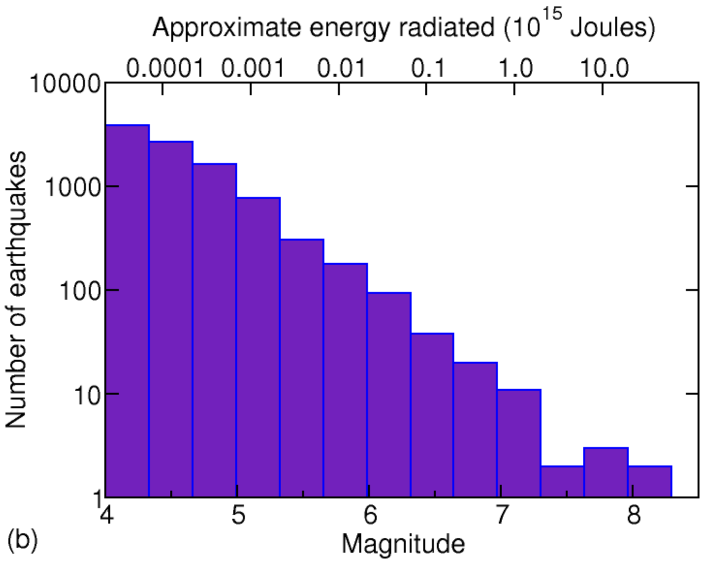
Earthquakes

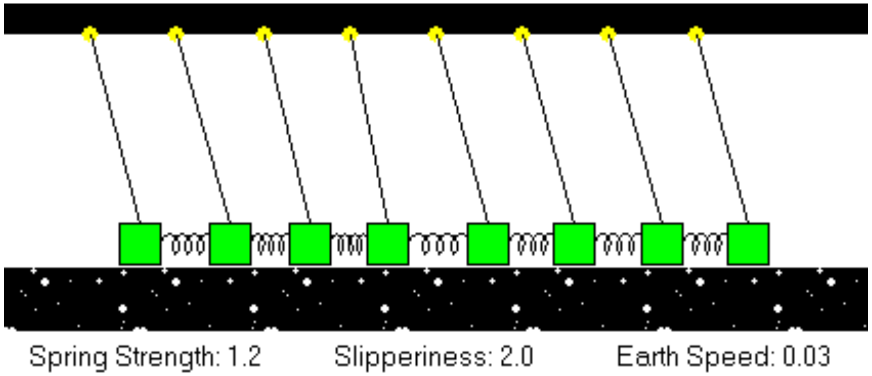
Spatially Extended Events of All Sizes



Earthquakes of Many Sizes: 1995

- Earthquakes of All Sizes
- Gutenberg-Richter Law:
  Probability ~ Size$^{-\text{Power}}$
- Simple Block-Spring Model
  - No disorder
  - Slow driving rate (cm/year)





Burridge-Knopoff (Carlson & Langer)

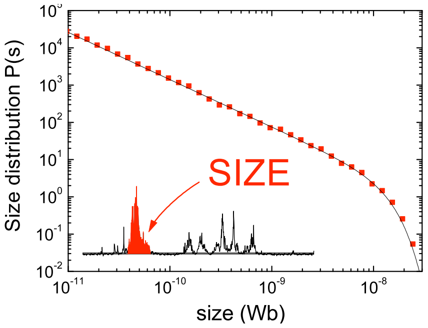
http://simscience.org/crackling/Advanced/Earthquakes/EarthquakeSimulation.html

Crackling noise

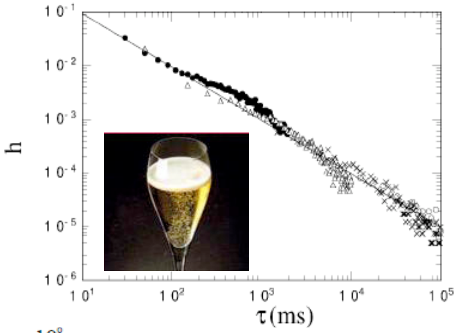
Discrete crackles span enormous range of sizes. Should be comprehensible; *scaling theory*.

Analogy with hydrodynamics:
Molecules don’t matter for Navier-Stokes fluid flow
Microscopics won’t matter for crackling

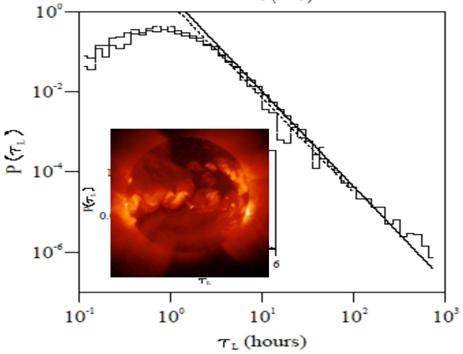
Magnets



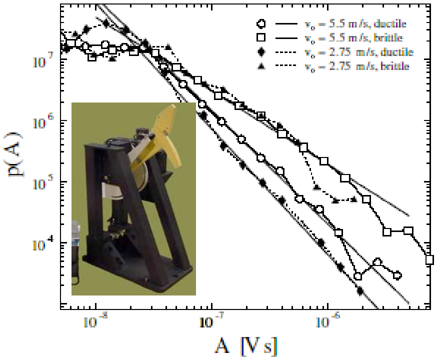
Foams

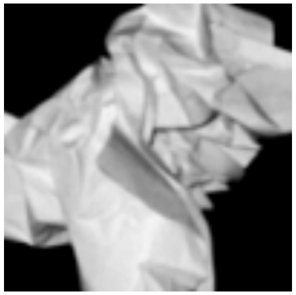


Solar Flares



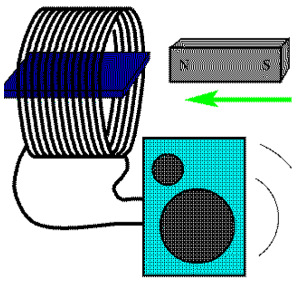
Fracture





Paper Crumpling

Rice Krispies

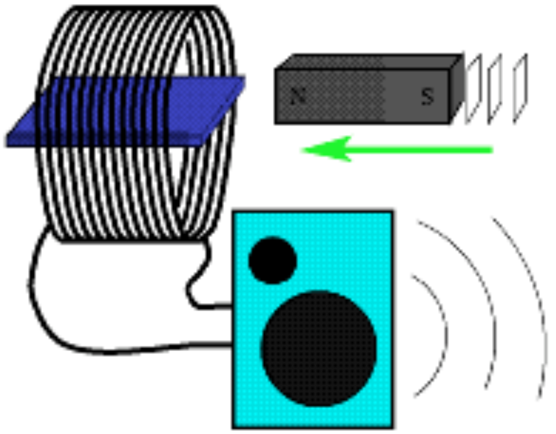


Magnets

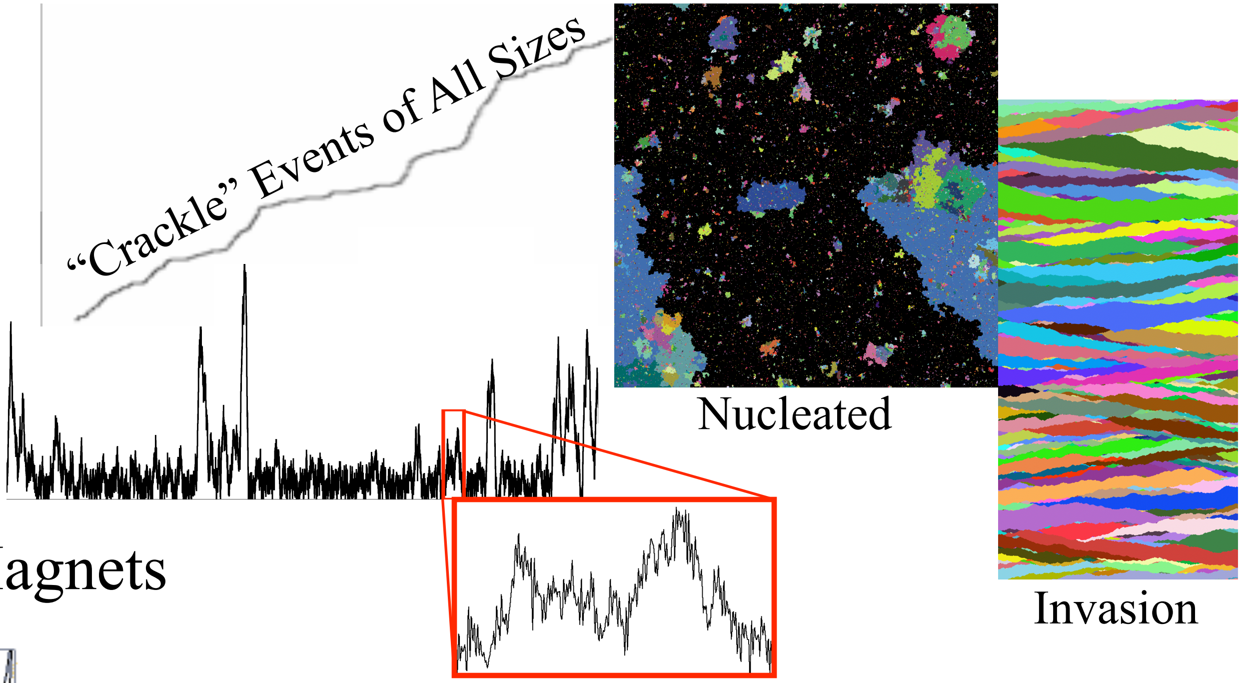
Tearing Paper

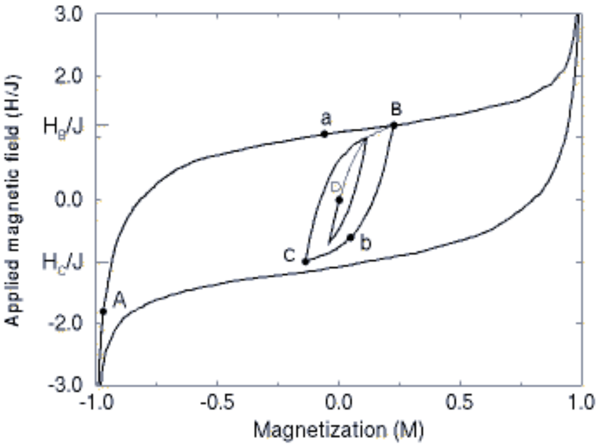
Magnetic Barkhausen Noise

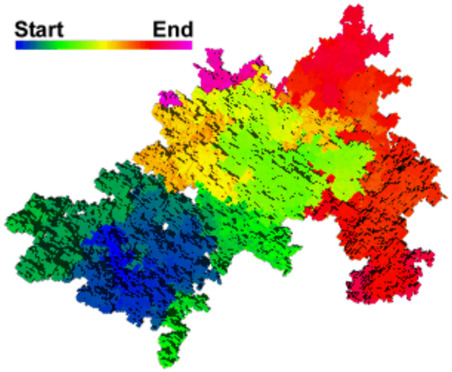
Events of All Sizes, Structure on All Scales



Barkhausen Noise in Magnets



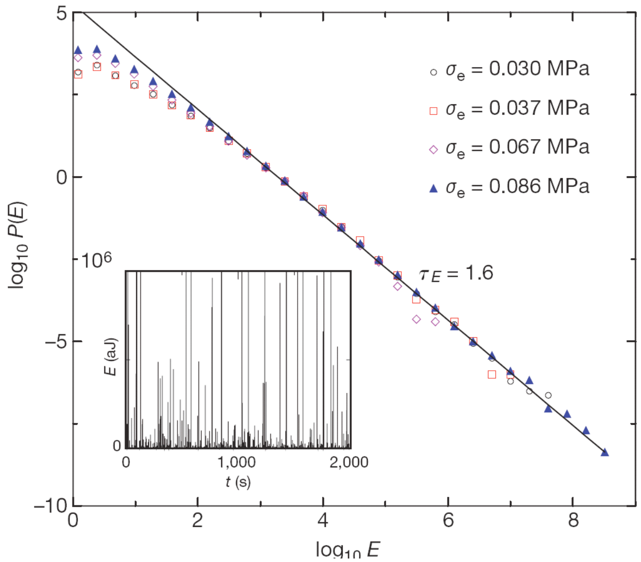




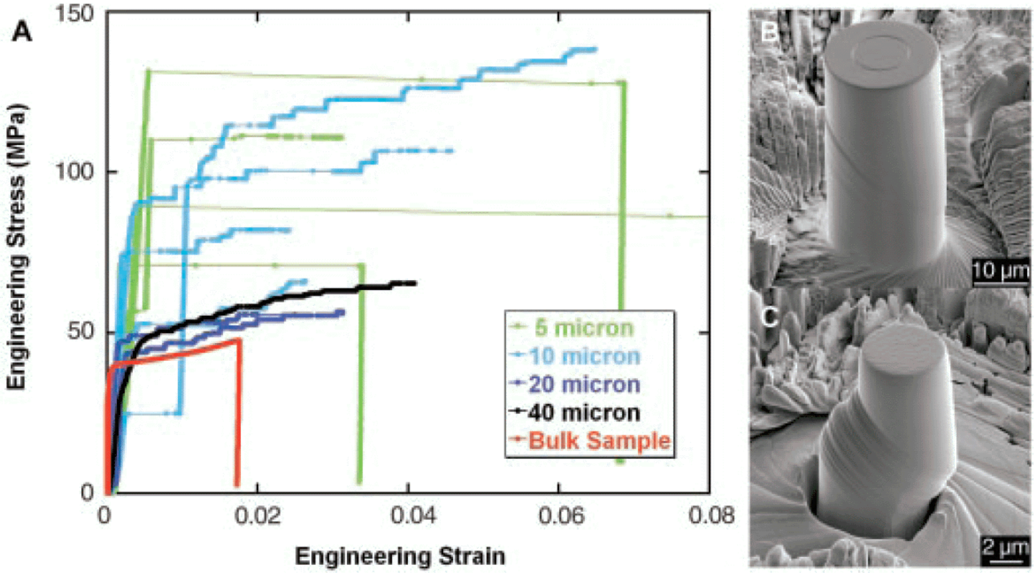
Magnetic
Avalanches
Fractal
in Time and
Space

Plasticity

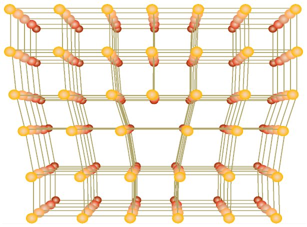
Dislocation avalanches when bending forks



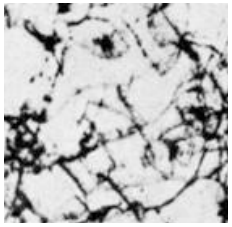
Avalanches in Ice
(Miguel et al.)



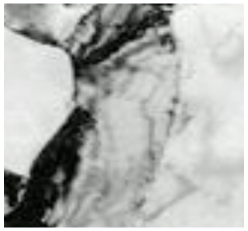
Avalanches in Nickel Micropillars
(Uchic et al.)



Dislocation



Tangle

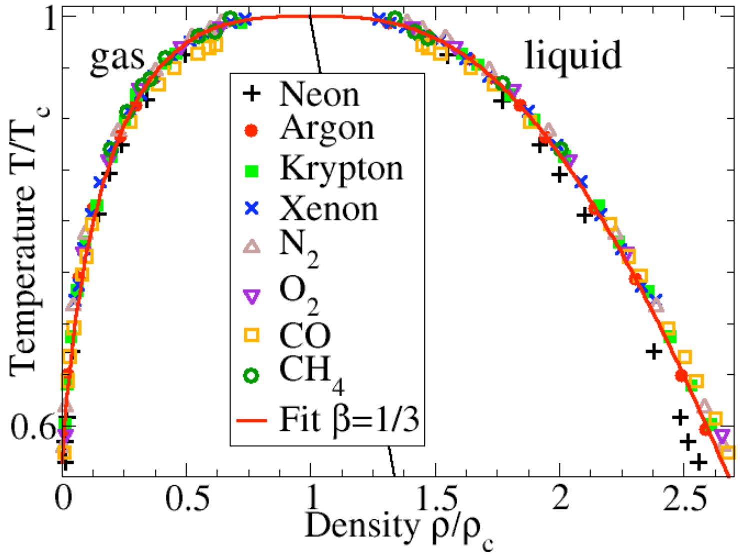


Structure

Ice crackles when it is squeezed
So, surprisingly, do metals

Universality: Shared Critical Behavior

Ising Model and Liquid-Gas Critical Point



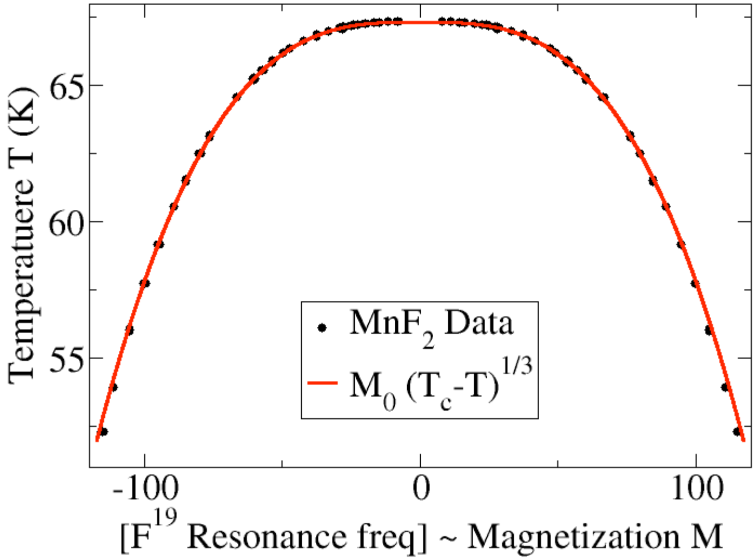
Liquid-Gas Critical Point

$$\rho - \rho_c \sim (T_c - T)^{\beta}$$

$$\rho^{Ar}(T) = A\,\rho^{CO}(BT)$$

Same critical exponent

β=0.332!



Ising Critical Point

$$M(T) \sim (T_c - T)^{\beta}$$

$$\rho^{Ar}(T) = A(M(BT),T)$$

Universality: Same Behavior up to Change in Coordinates

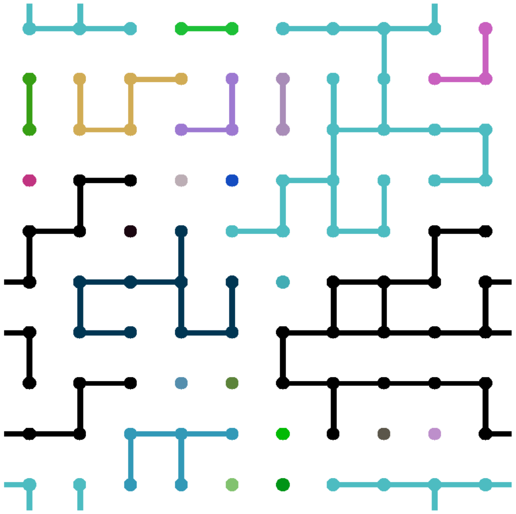
$$A(M,T) = a_1 M + a_2 + a_3 T + \textit{(other singular terms)}$$

Nonanalytic behavior at critical point (not parabolic top)

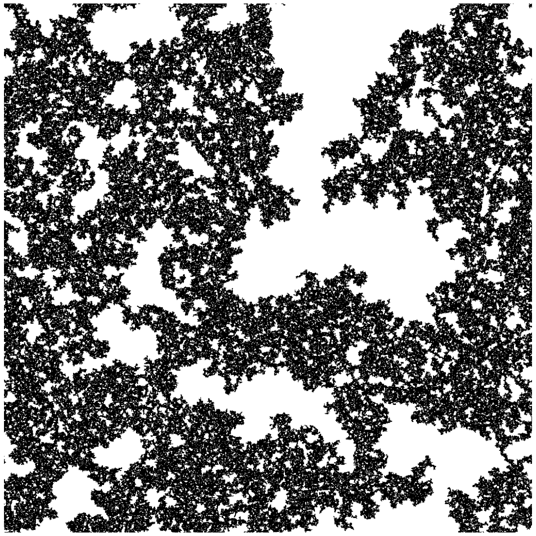
All power-law singularities (χ, c_v, ξ) are shared by magnets, liquid/gas

Microscopic Details Irrelevant

Universality in Percolation

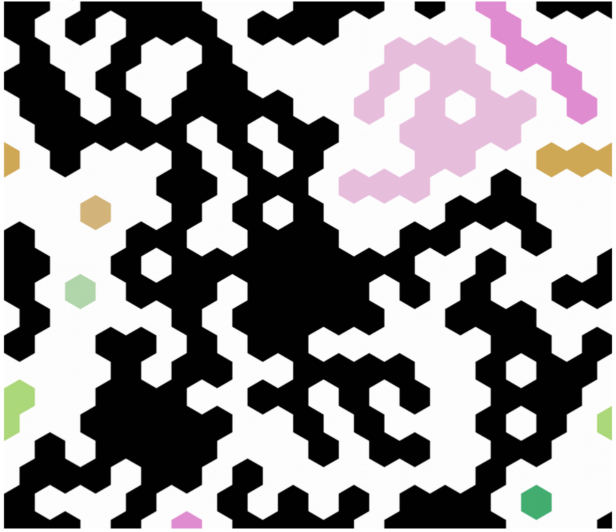


Bond Percolation

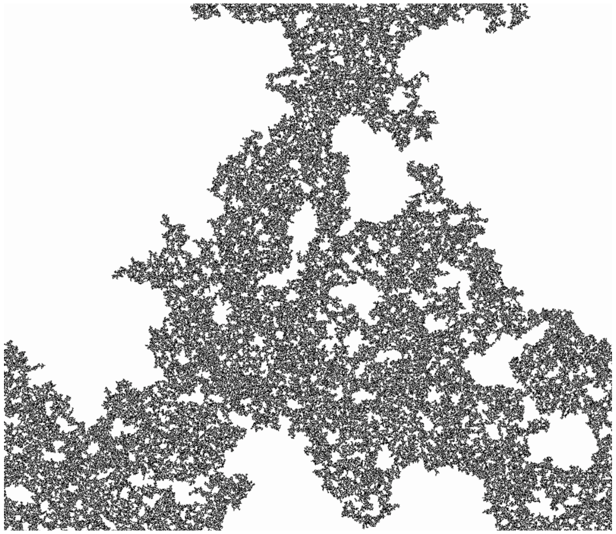


Site, bond percolation look the same for big clusters!

Statistical morphology of critical point independent of microscopic details: depends only on dimension of space, type of transition (*universality class*). (Note site percolation lighter: overall scale of order parameter non-universal)



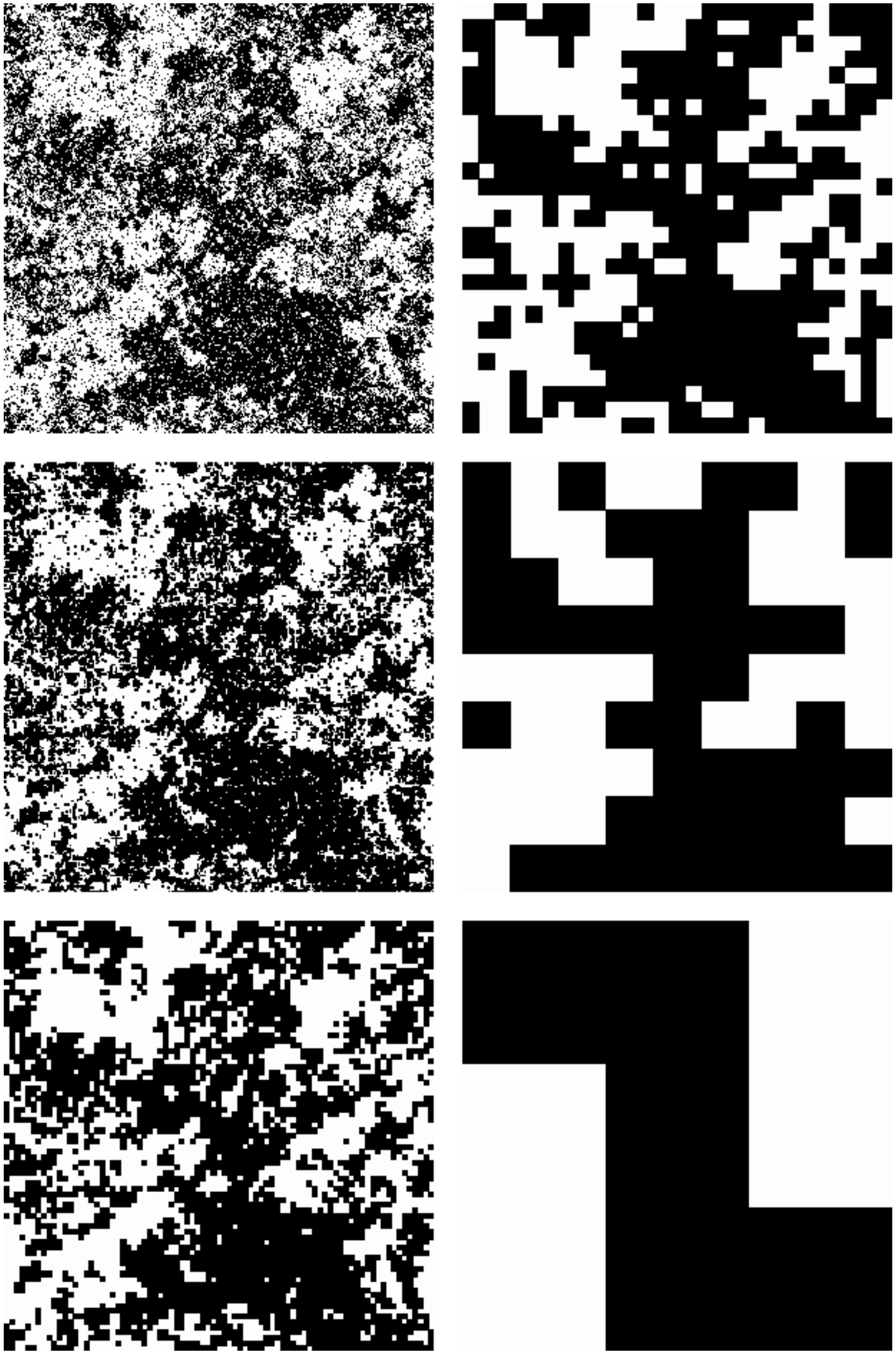
Site Percolation



Coarse Graining

Remove microscopic details

- ‘Continuum limit’ – average over details in small regions, get effective laws for coarser system
- Example: majority-rule block-spin transformation (3x3 blocks)
- Renormalization group: find effective block-spin free energy: new interactions from old by tracing over microscopic variables

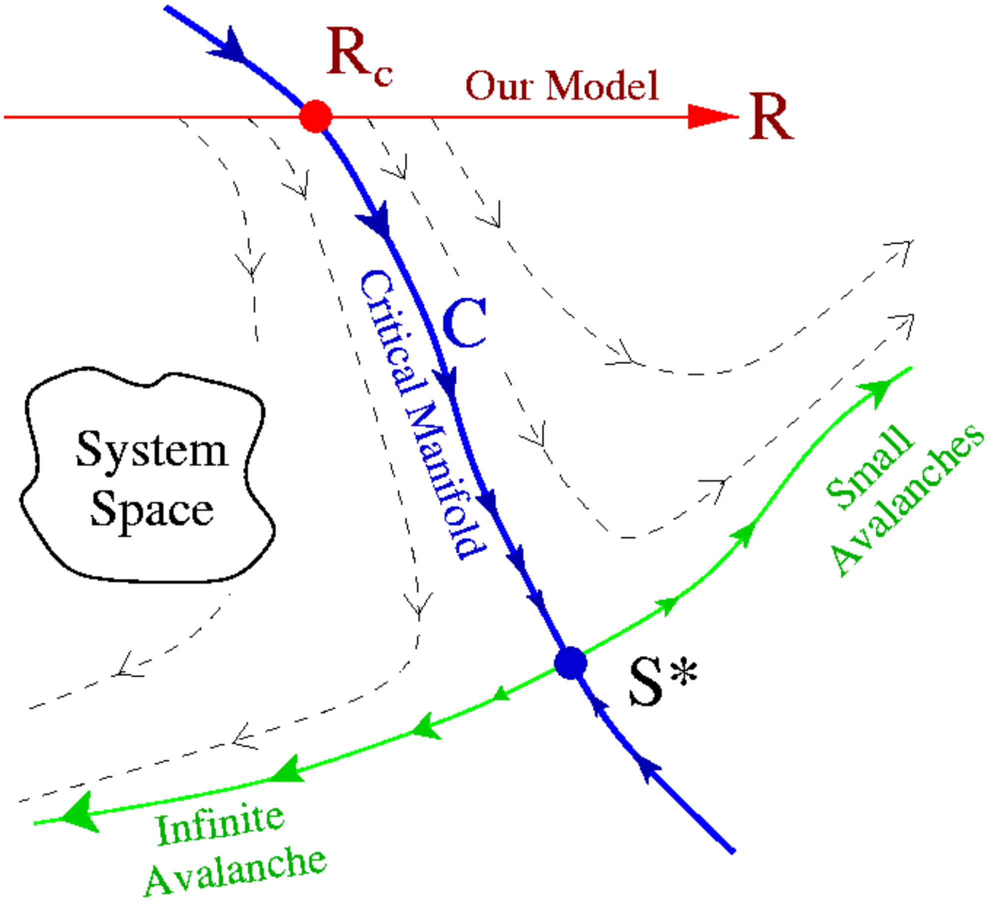


The Renormalization Group

Why Universal? Fixed Point under Coarse Graining

Renormalization Group

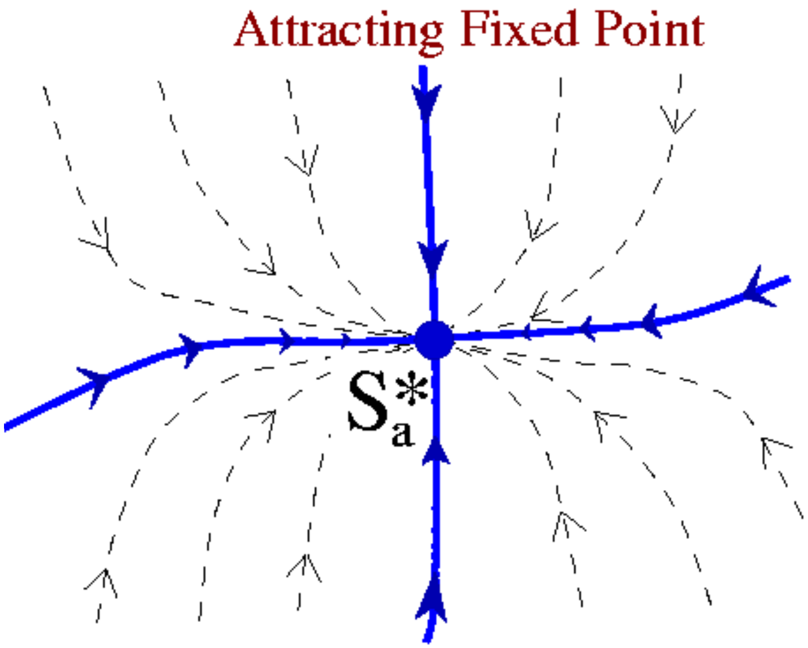
- Not a group
- *Renormalized* parameters (electron charge from QED)
- Effect of coarse-graining (shrink system, remove short length DOF)
- Fixed point **S*** ***self-similar*** (coarse-grains to self)
- Critical points flow to **S*** ***Universality***
- Many methods (technical) real-space, ε-expansion, Monte Carlo, …
- Critical exponents from linearization near fixed point

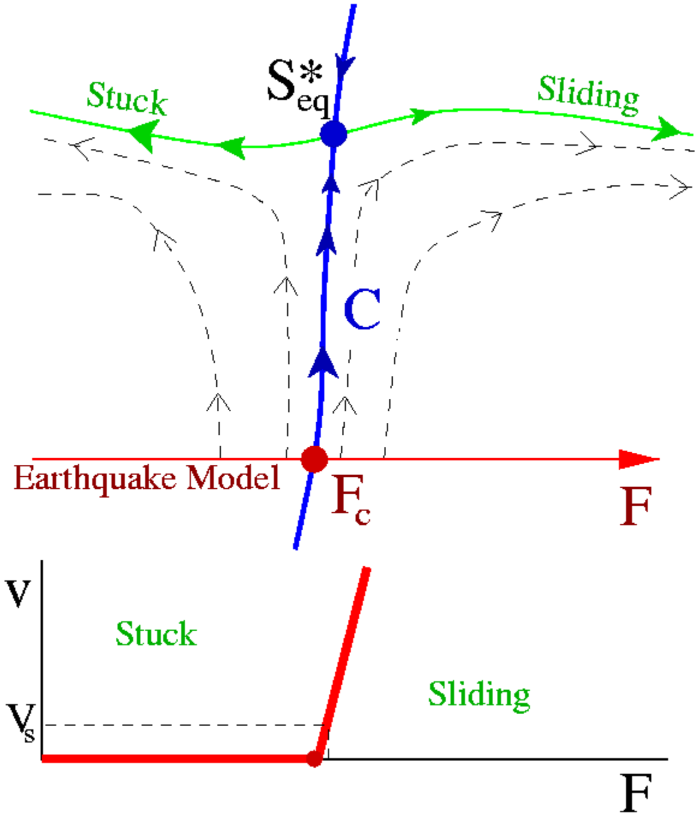


System Space Flows Under Coarse-Graining

Spontaneous Criticality

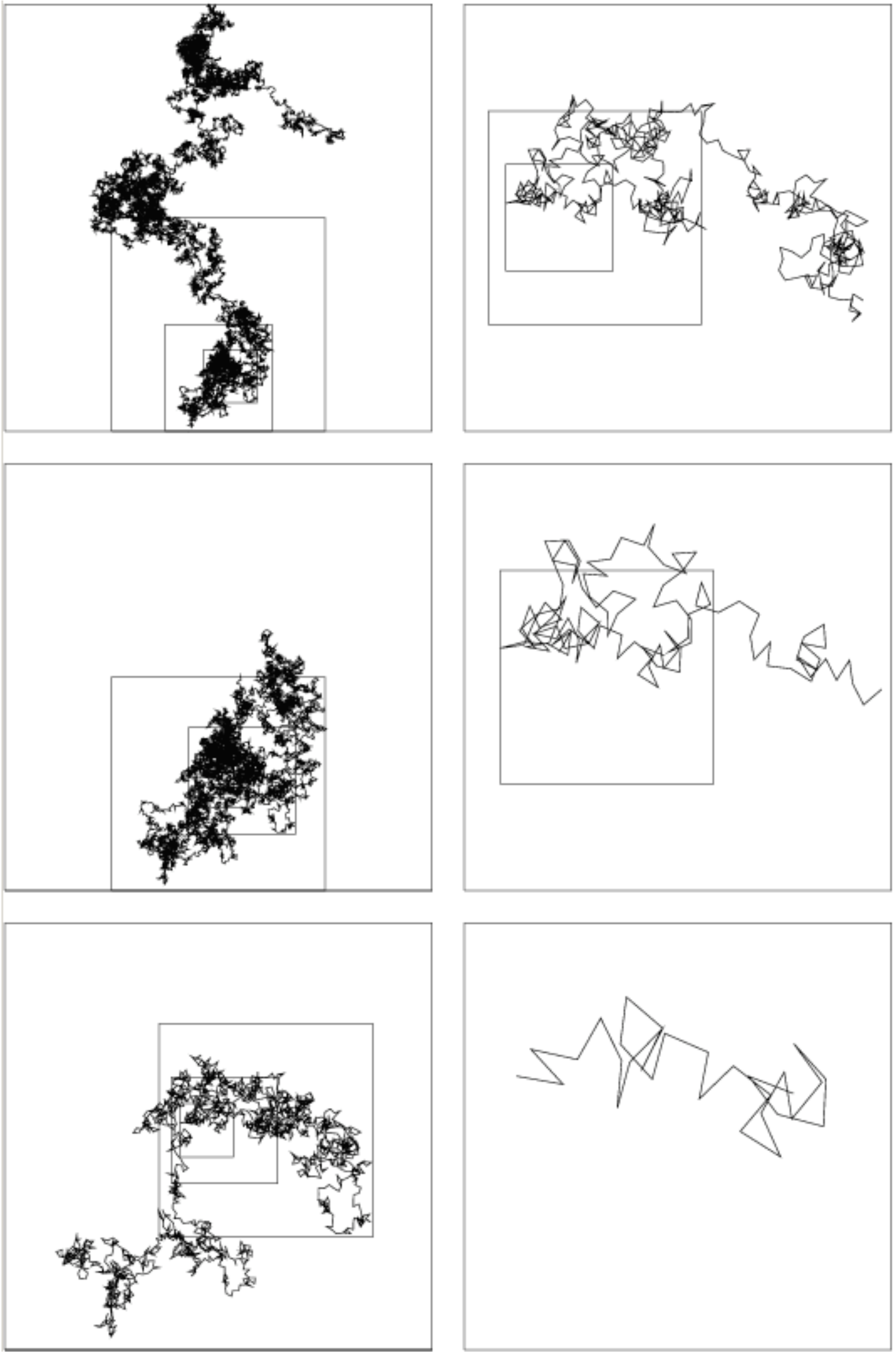
Generic Scale Invariance; Self-Organized Criticality





Attracting Fixed Point: Phases!
Sometimes still fluctuations
(Polymers, random walks,
surface growth)

Slow Driving / Inhomogeneous /
Long-range Forces
Drives to Critical Point
(Earthquakes, Sandpiles, Front
Propagation, Forest Fires…)



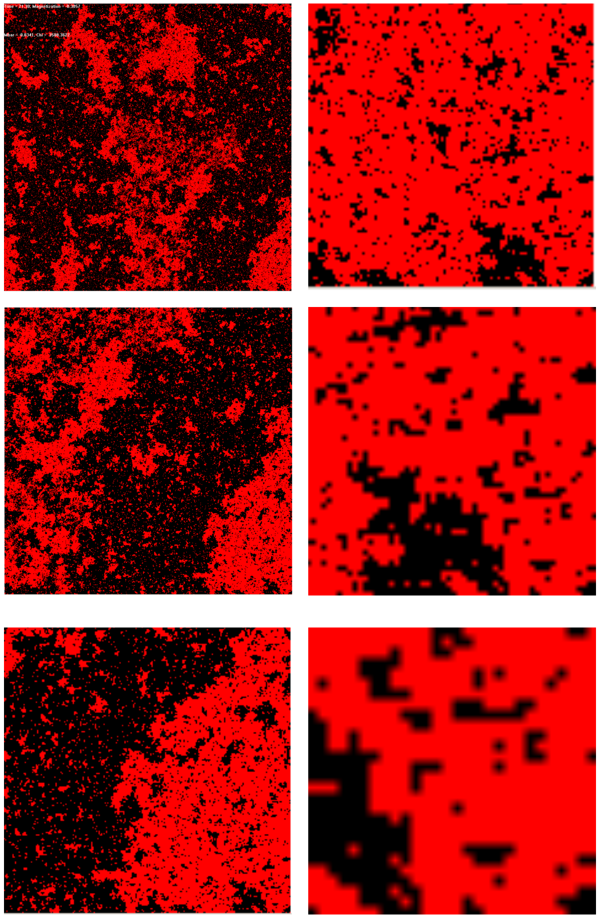
Random Walks: Self Similarity

¼ of an N-step walk is (statistically) like shrinking by ½

- Endpoint ~ ($\sqrt{N}$ a) half as far
- Fractal: self similar
- Mass ~ radius$^{\text{fractal dimension}}$
- Random walk dimension = 2

Self-Similarity

Self-Universality on Different Scales



Ising Model at T_c

Fixed point **S*** maps onto *itself*, at a longer scale: *self-similar*. Models cross **C** at critical point T_c, flow to **S***: also self-similar.

Self-similarity → Power Laws

Expand rulers by $B=(1+\varepsilon)$;

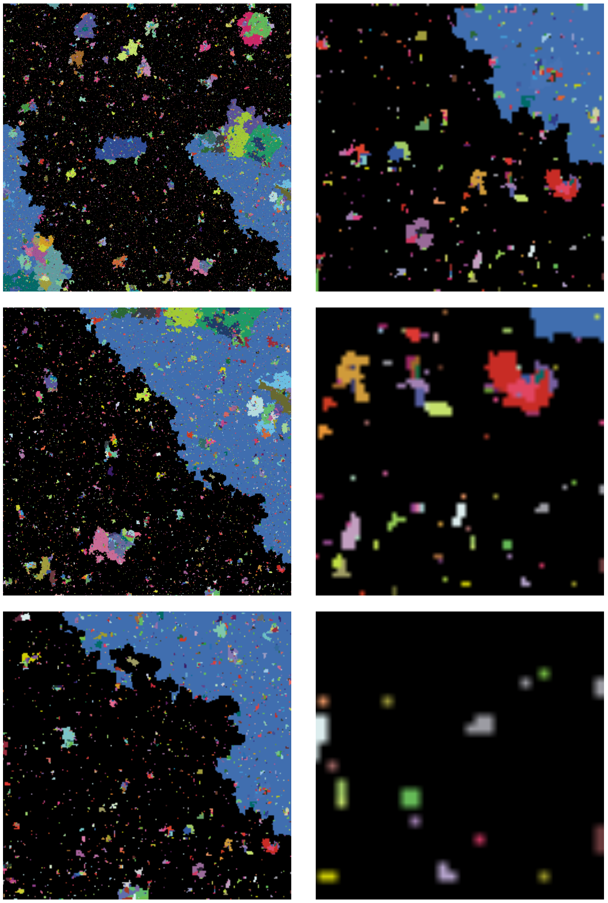
Avalanche size distribution

$$D'[S'] = A\, D[C\, S']$$

$$=(1+a\varepsilon)\, D[(1+c\varepsilon)\, S')]$$

$$a\, D = -c\, S\, dD/dS$$

$$D[S] = D_0\, S^{-a/c}$$



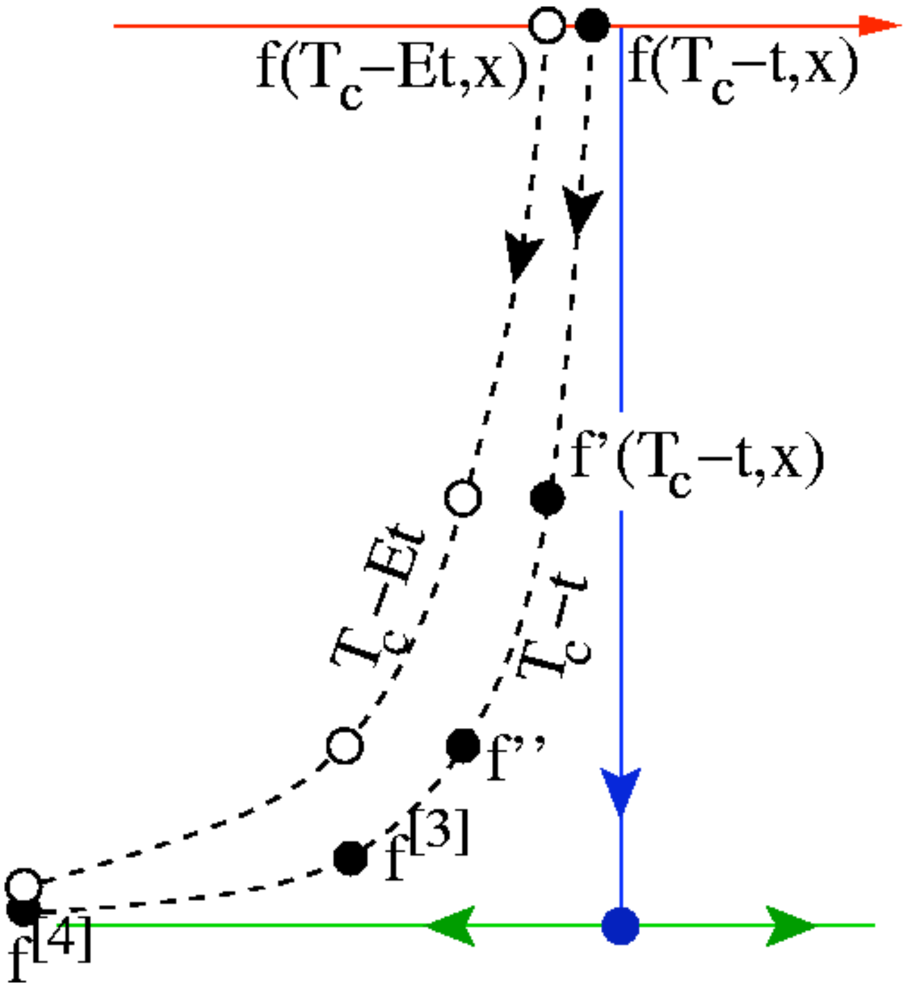
Hysteresis Model at R_c

Universal critical exponents $c=d_f=1/\sigma\nu$, $a/c=\tau$: D_0 system dependent

Ising Correlation $C(x) \sim x^{-(d-2+\eta)}$ at T_c, random walk $x \sim t^{1/2}$

Scaling Near Criticality

Self-Universality on Different Scales



RG Flow near Critical Point.

Two points that flow toward one another must be similar on long length scales.

$$f^{[4]}(T_c-t,\, x) = f^{[3]}(T_c-Et,x)$$

so

$$f'(T_c-t,y)=Af(T_c-t,By) \sim f(T_c-Et,y)$$

at large y: the system is similar to itself at a different set of parameters.

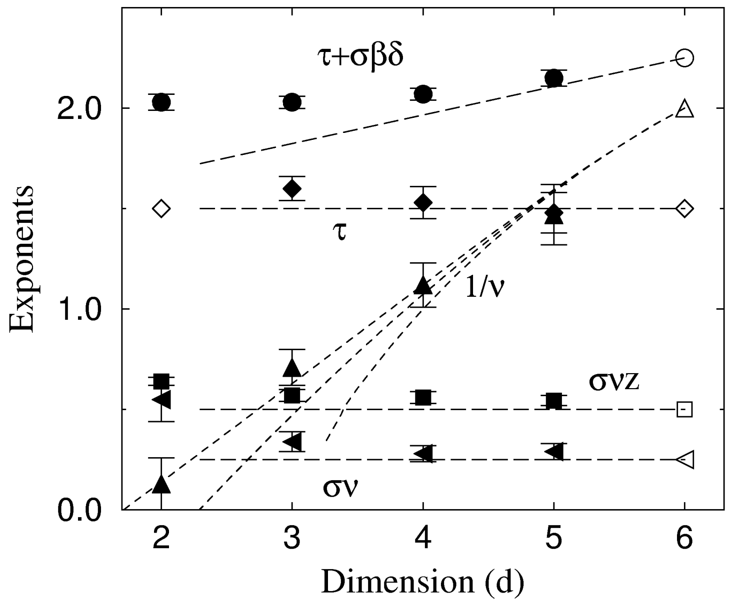
$$M'(T_c-t)=AM(T_c-t)=M(T_c-Et)$$

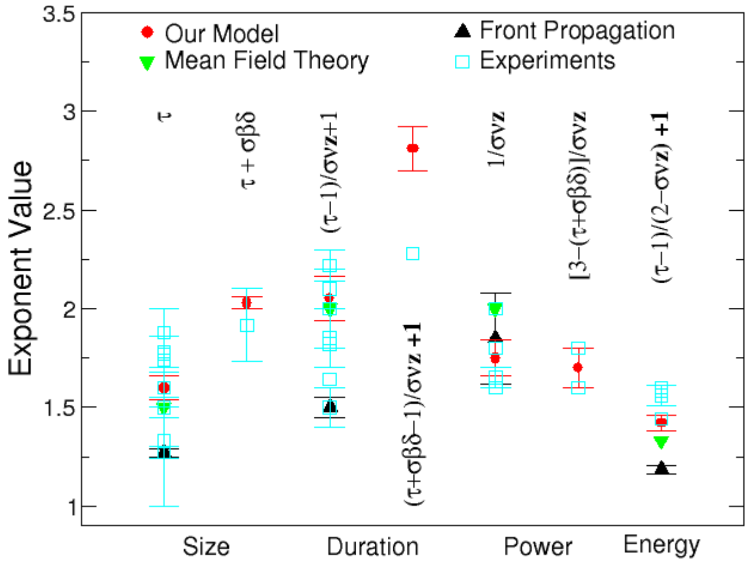
$$(1+\beta\varepsilon/\nu)\, M(T_c-t)=M(T_c-t(1+\varepsilon/\nu))$$

$$\ldots\ M \sim (T_c-t)^{\beta} \sim t^{\beta}$$

Critical Exponents

Combinations of Greek Letters





σ: Maximum avalanche size $S_{max} \sim (R\text{-}R_c)^{-\sigma}$

ν: Correlation Length $\xi \sim (R\text{-}R_c)^{-\nu}$

τ: Probability of Avalanche $P(S, R_c, H_c) \sim S^{-\tau}$

$\bar{\tau} = \tau+\sigma\beta\delta$: Integrated Probability $P_{int}(S, R) \sim S^{-(\tau+\sigma\beta\delta)}$

$\sigma\nu$: Fractal Dimension $1/\sigma\nu$

z: Duration $T \sim \xi^{z} \sim (R\text{-}R_c)^{-\nu z}$

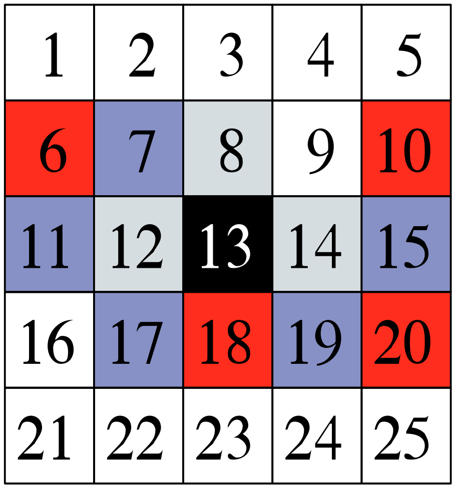
Hysteresis Model for Magnets

T=0 Driven Random-Field Ising Model

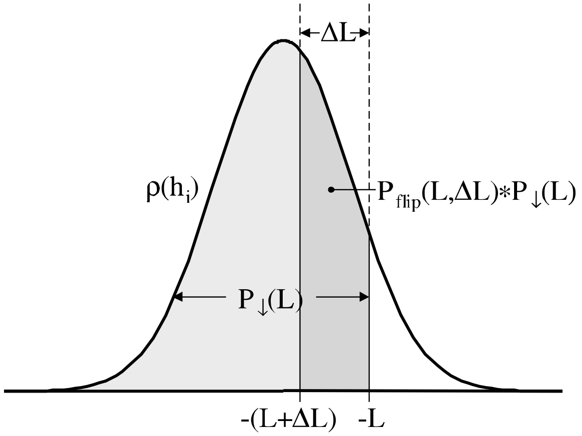
$$H=-\Sigma_{ij\ nn}\, J\, S_i\, S_j - \Sigma_i\, H\, S_i - h_i\, S_i$$

$S_i = \pm 1$, magnetic domain
J coupling between neighboring spins
H external applied field
h_i random field at site, dirt,
chosen from Gaussian width ***R***

Lattice



1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25



$P(h)$ = Gaussian RMS width ***R***

Dynamics

- Start all spins down, $H=-\infty$
- Increase field slowly
- Spin flips when pushed over
- Initial spin 13 pushed by H
- Pushes neighbors: avalanche!

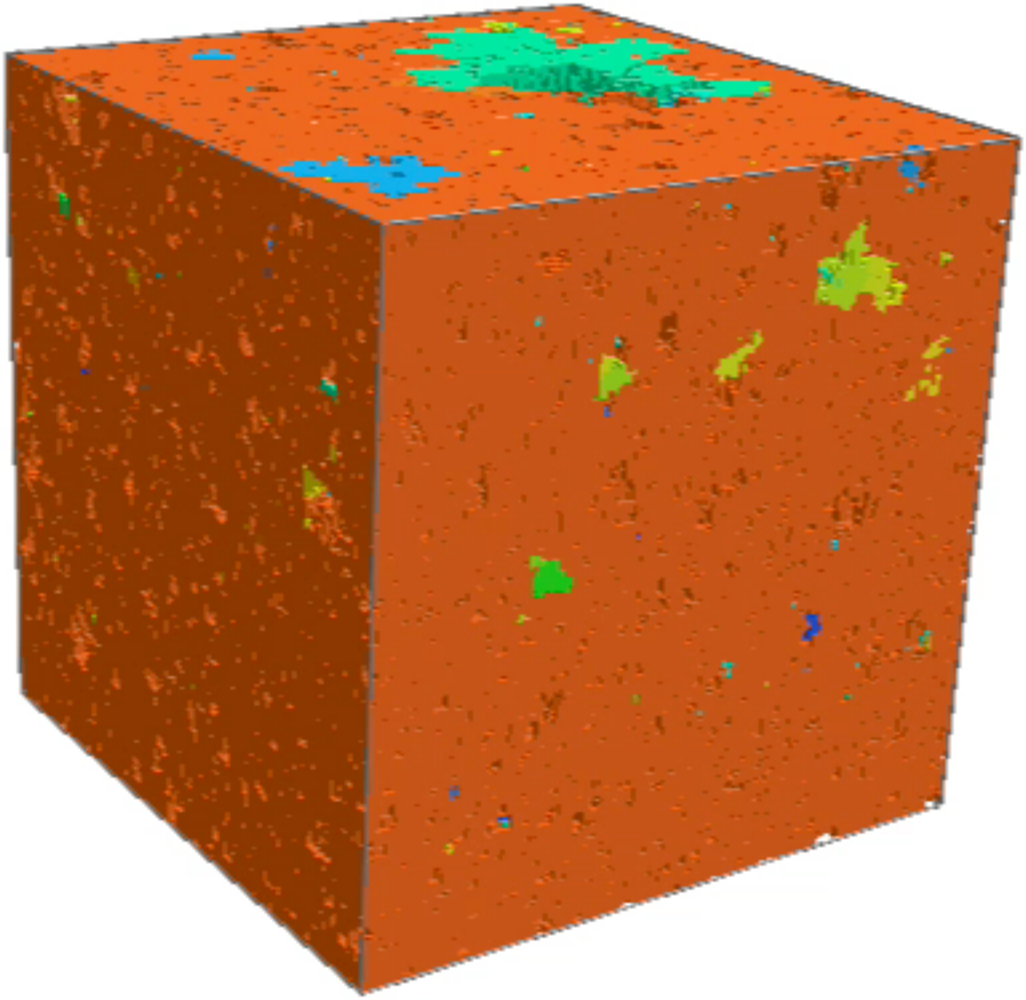
$V(t)$ = number of spins in shell t

Simulation at the critical disorder

(Chris Pelkie)

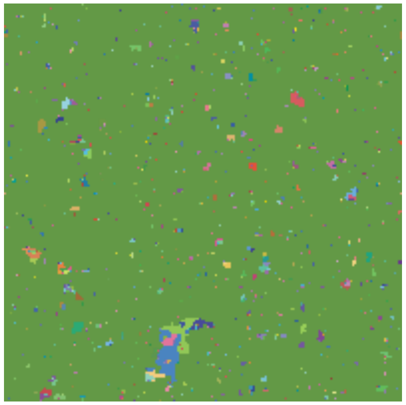
System tuned to special
"critical" disorder *R*

- Avalanches of all scales
- Early small avalanches, growing in size
- "Infinite" (red) avalanche, large jump in magnetization
- Small final avalanches fill in gaps

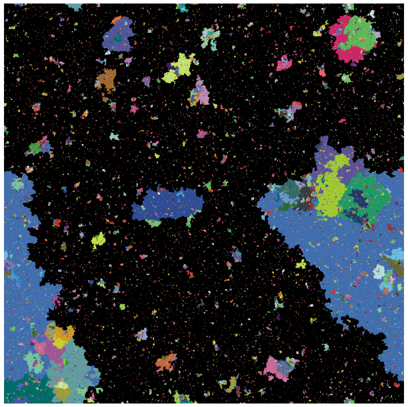


Phase Transition in Nucleated Hysteresis

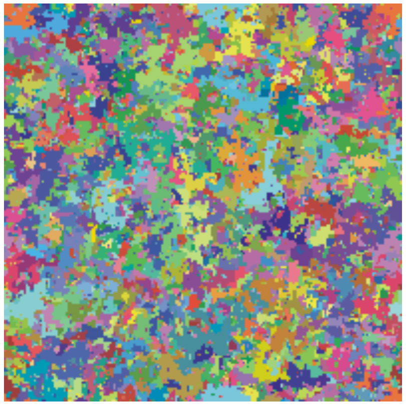
Critical Disorder: First Infinite Avalanche



R=2



$R=R_c=2.16$



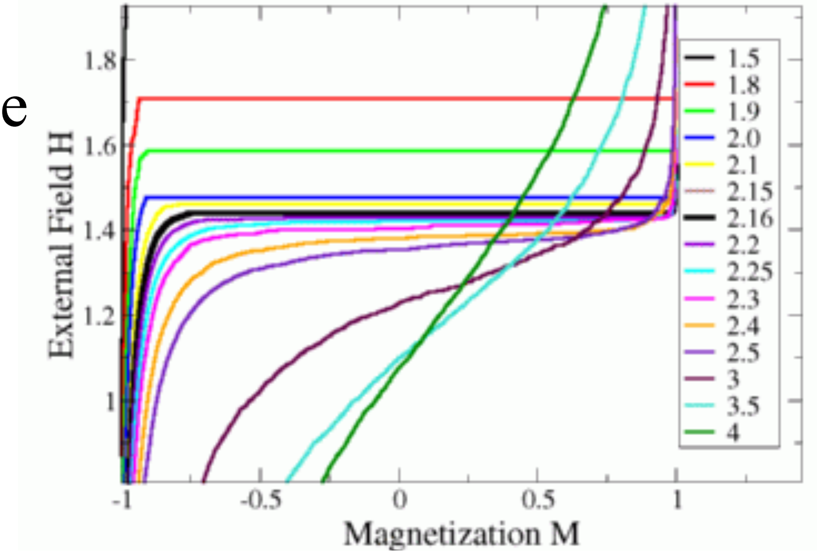
R=2.5

Small Disorder

Neighbors Dominate

One Big Avalanche

(First Spin Triggers All)



Large Disorder

Dirt Dominates

Many Small Avalanches

(Each Spin to Itself)

At R_c,

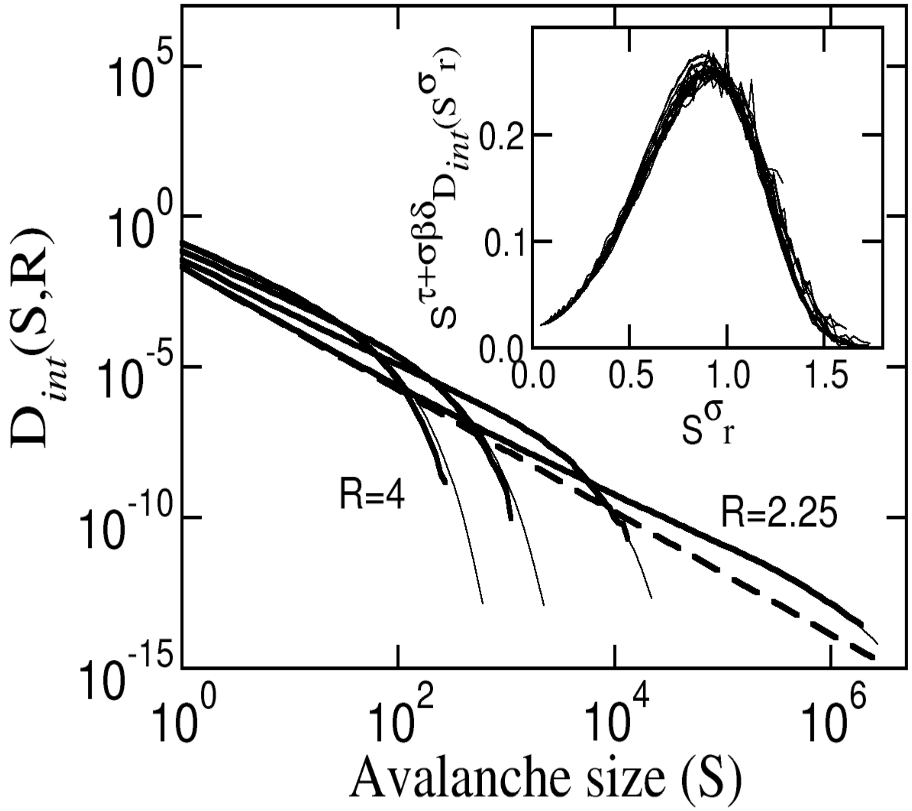
$M-M_c \sim (H-H_c)^{1/\delta}$

Transition in Shape of Hysteresis Loop

What happens away from R_c?

Scaling Functions

Self-Universality away from Criticality



- Avalanche Size Distribution $D(S)$
- At R_c get Power Law

  $D(S) \sim S^{-\bar{\tau}} \sim S^{-(\tau+\sigma\beta\delta)}$
- Big ones cut off at $(R-R_c)^{-\sigma}$
- Scale invariance: write as power law times function of one fewer variable …

$$D[S,R] = S^{-(\tau+\sigma\beta\delta)}\, \mathcal{D}[S/(R-R_c)^{-\sigma}]$$

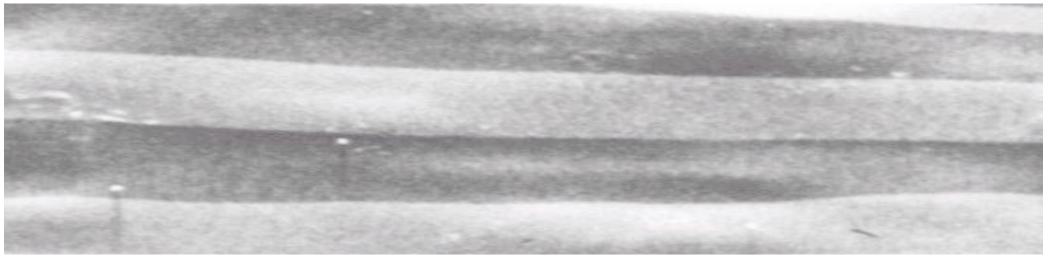
Universal Scaling Function $\mathcal{D}$

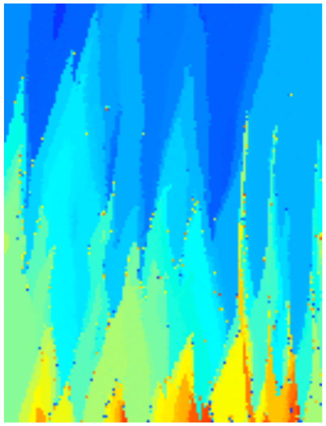
Scaling Collapse: Plot $S^{(\tau+\sigma\beta\delta)} D[S,R]$ vs. $S/(R-R_c)^{-\sigma}$, measure $\mathcal{D}$ (inset)

$$M(H,T)=(T_c-T)^\beta \mathcal{M}(H/(T_c-T)^{\beta\delta});\; C(x,t,T)=x^{-(2-d+\eta)}\, \mathcal{C}(x/|T-T_c|^{-\nu},t/|T-T_c|^{-\xi\nu})$$

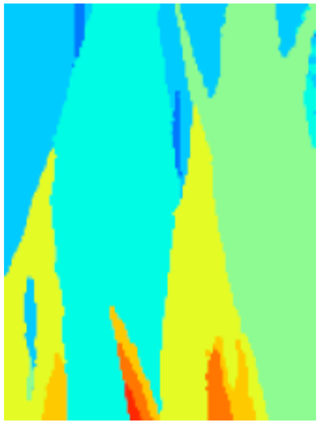
Real Barkhausen Noise

Motion of single fronts (Robbins, Fisher, Bouchaud)



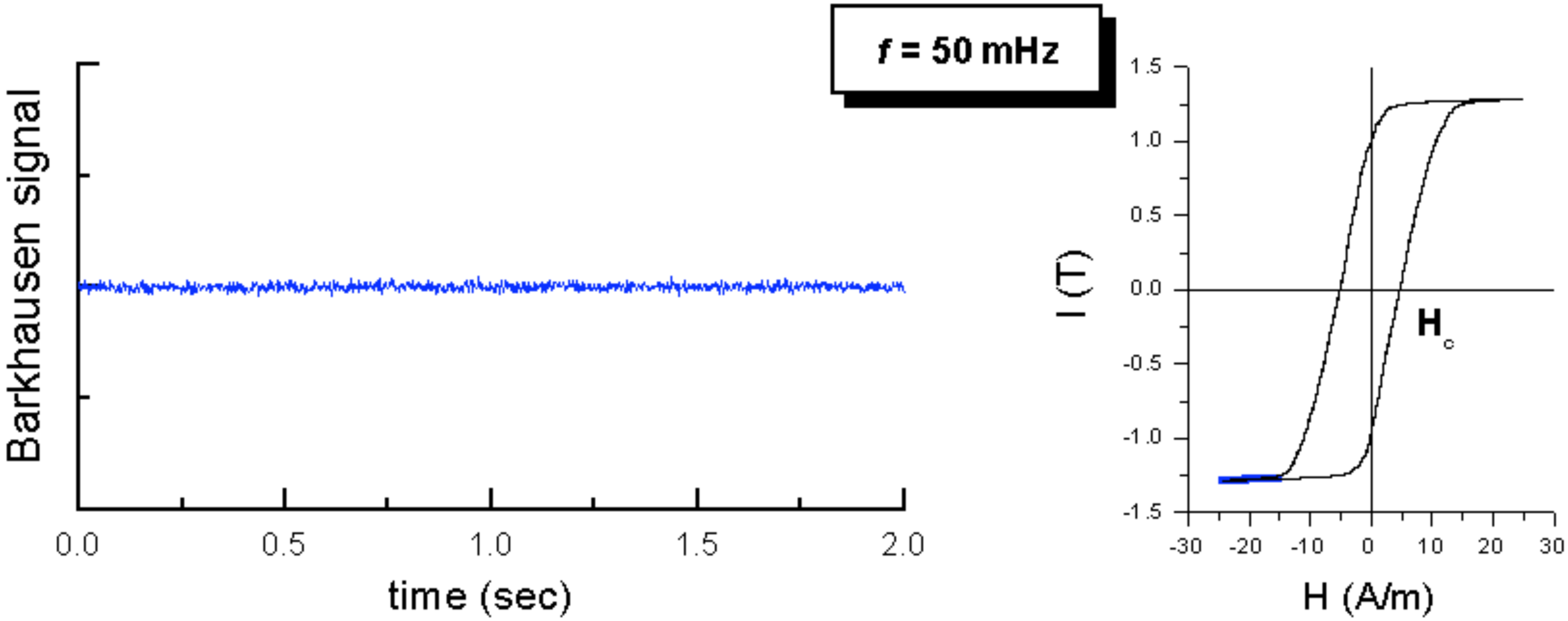


880×660 μm



440×330 μm

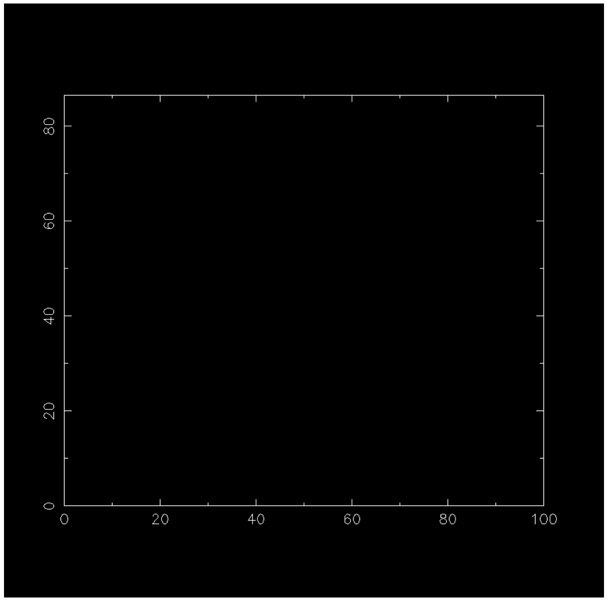
Gianfranco Durin



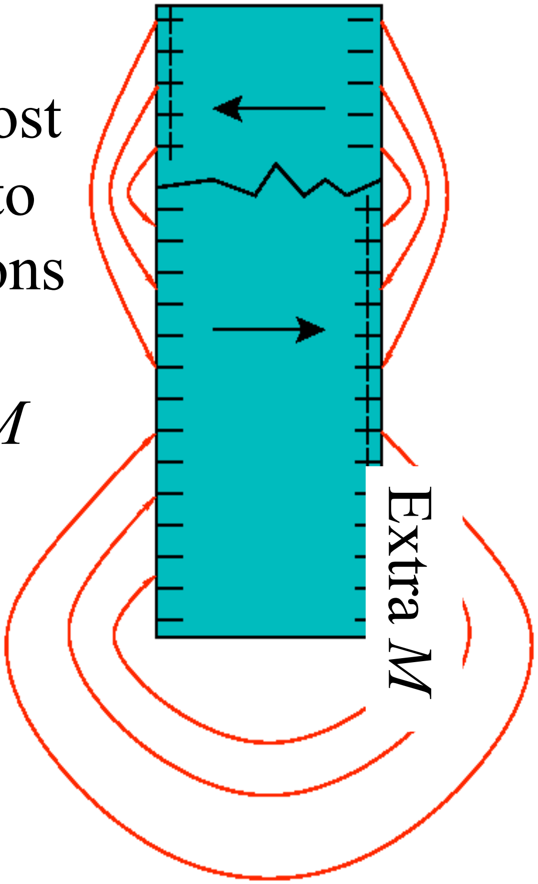
Demagnetizing Field

How magnets self-organize to front depinning point

Once the external field depins a front, why does it ever stop moving?

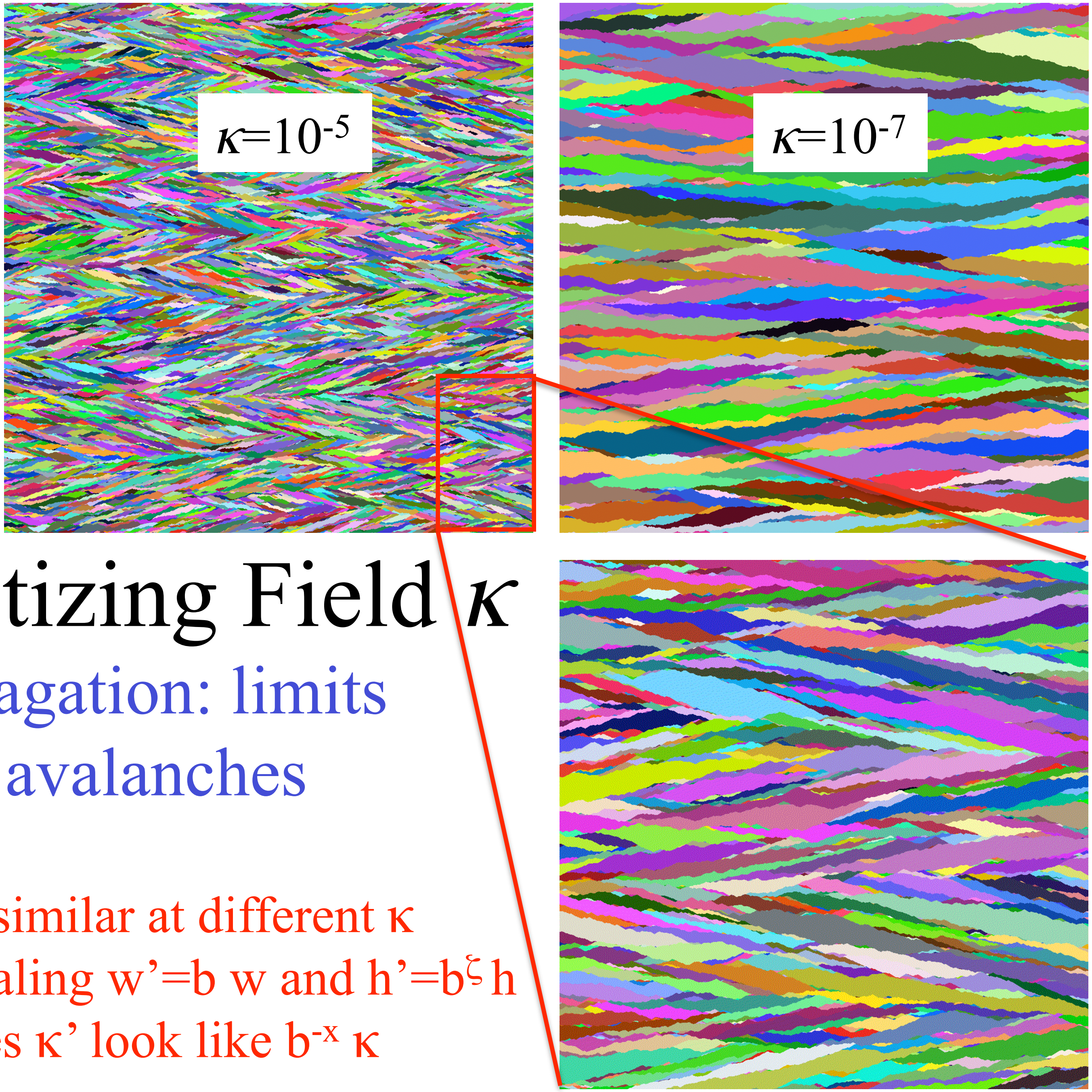


Long-range dipolar fields (+/- = N/S) cost energy mostly due to non-canceling regions (hence ~ κM, net magnetization M × demagnetization factor κ).



κ acts as much like T-T_c, a *relevant* perturbation that cuts off the largest avalanches

YJ Chen,
Others



Demagnetizing Field κ

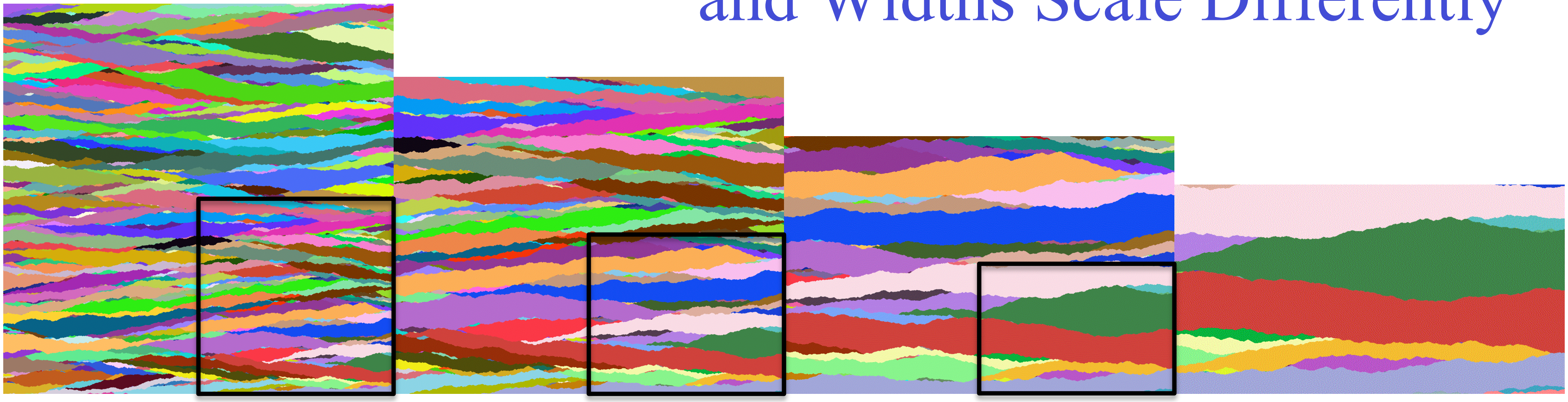
Front propagation: limits sizes of avalanches

Self-similar at different κ
Rescaling $w'=b\,w$ and $h'=b^{\xi}\,h$
makes κ' look like $b^{-x}\,\kappa$

YJ Chen,
Others

Self-Affine

Front propagation: Heights and Widths Scale Differently



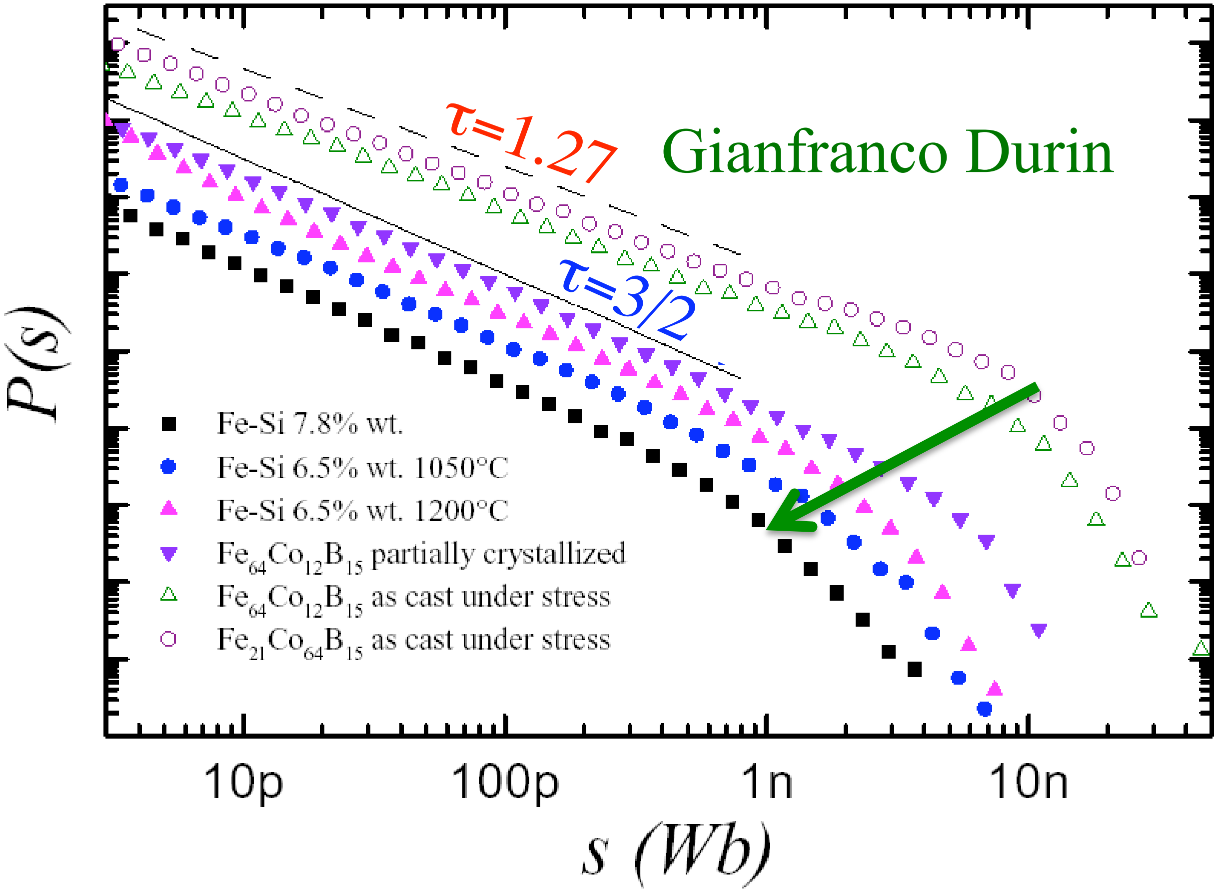
Cut bottom left-hand quarter
Rescale widths by 2, heights by 2^{ζ}
Effective lower demagnetizing field:
    larger avalanches
Fronts appear statistically similar: self-affine

Barkhausen Noise Size Distributions

3D Universality classes

Avalanche size distributions (and other critical exponents) cluster into two families. One is the *front propagation model*, the other is a *mean-field theory* due to long-range forces.

(Our model doesn't describe any of the experiments.)



Cutoff $S_{max} \sim \kappa^{\sigma\kappa}$

- Different systems, same exponents
- Experiment and theory, same exponents

Universality!

Beyond Power Laws

Universal Scaling Functions

Functions of one variable become power laws at critical points

$$P(S) = b^x P(S/b^{df}) = b^{nx} P(S/b^{ndf}) = \ldots = s^{-\tau}$$

Functions of *N* variables become power laws times *universal* functions of *N-1* scaling variables

$$P(S, H, W \,|\, \kappa) = b^y P(S/b^{df}, H/b^{\xi}, W/b \,|\, \kappa/b^{x_k})$$

$$= \ldots = s^{-\omega} \mathcal{P}(H^{1+\xi}/S, W^{1+1/\xi}/S, \kappa^{\sigma_\kappa}/S)$$

Universal scaling functions forms:

- avalanche shape, $T^{1/\sigma\nu z-1} \mathcal{V}(t/T)$
- avalanche energy, $S^2 \mathcal{E}(\omega^{1/\sigma\nu z} S)$
- avalanche size/duration,

$$S^{-\tau} \mathcal{P}_{\pm} (S/\kappa^{\sigma_\kappa})$$

Ising model, other critical points

- magnetization $r^{\beta} \mathcal{M}_{\pm}(h/r^{\beta\delta})$
- correlation length

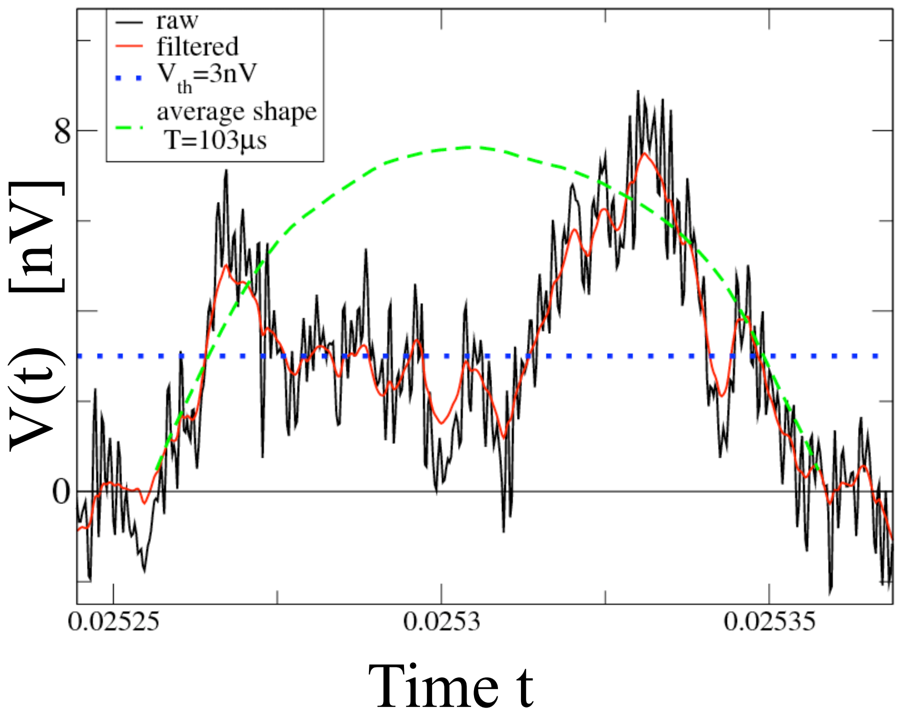
$$\xi(T,H) = t^{-\nu} \mathcal{Y}\pm (h/t^{\beta\delta})$$

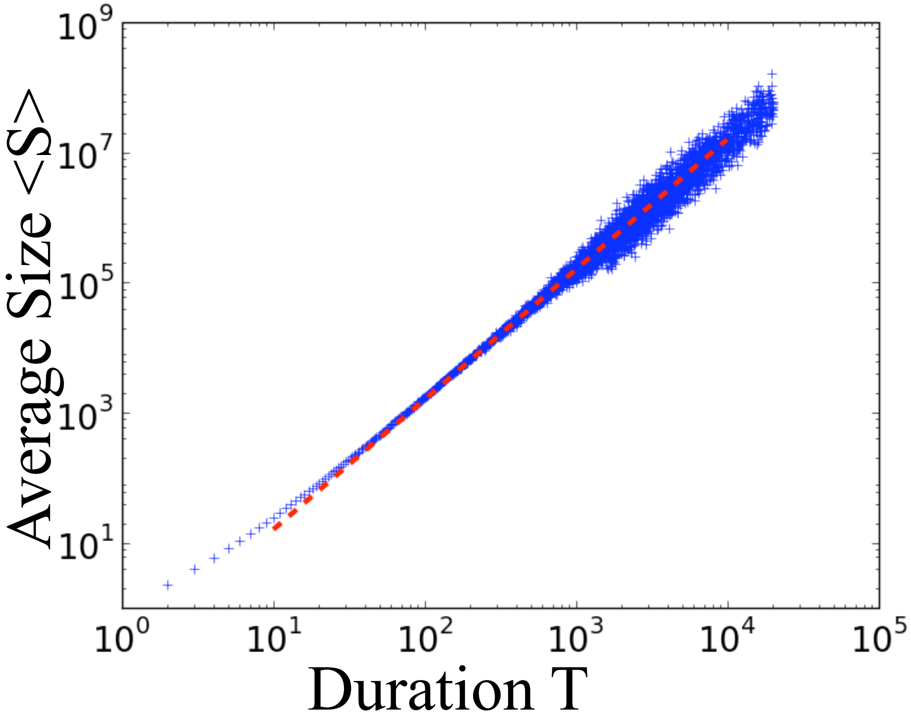
- finite size scaling …

Avalanche Temporal Structure

Stefanos
Papanikolaou,
Others

Duration scaling, fractals, average shapes





Hierarchical structure in time
avalanche almost stops
many times

Average shape for given duration: dashed green line

Average size S grows with duration

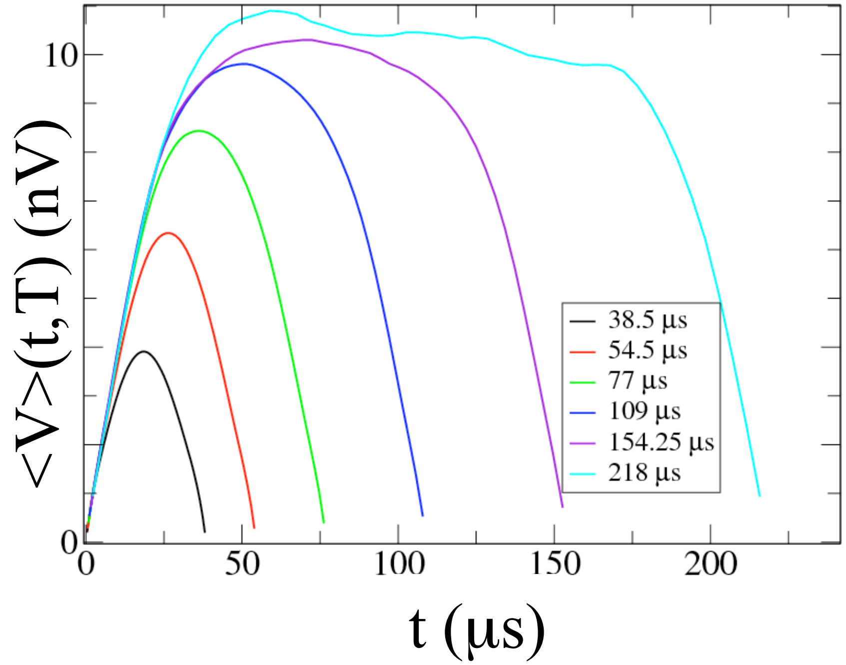
$$S \sim \xi^{\sigma\nu} \sim T^{\sigma\nu z}$$

Another power law…

Scaling Collapses

Universal *functions*

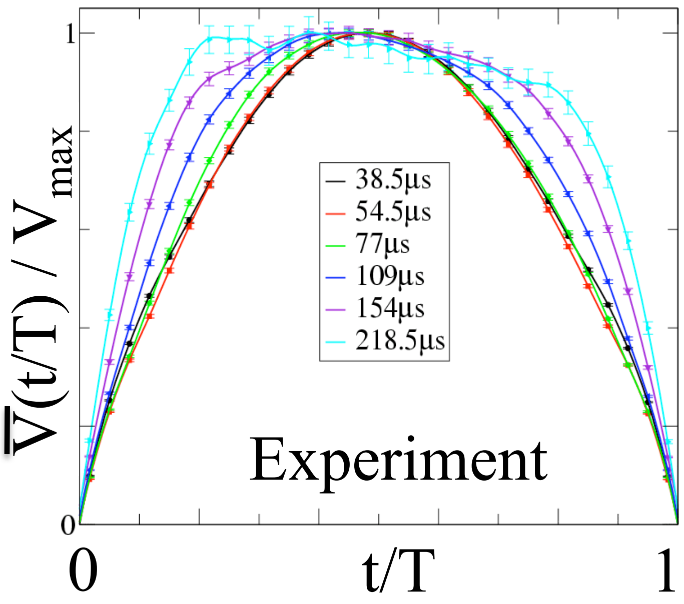
Average <V>(t,T) over avalanches of duration T

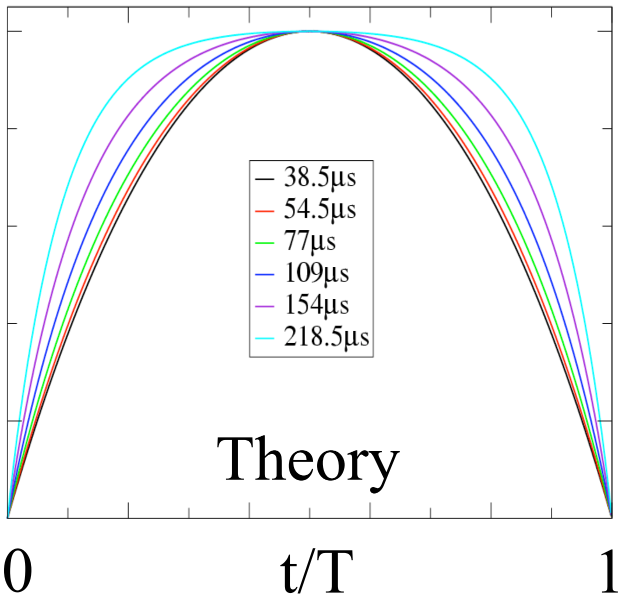


Scaling Collapse

- Divide variables by scale (time T, voltage V_{max})
- Universal scaling form (parabola MF)
- Demagnetizing field crossover flattening

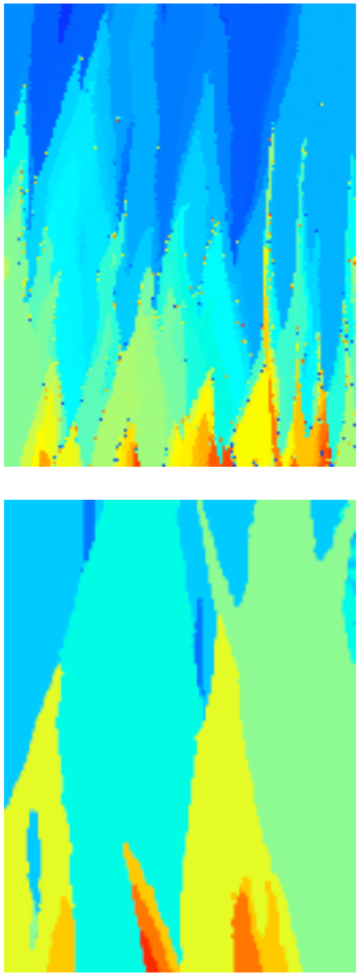
Scaling away from criticality?





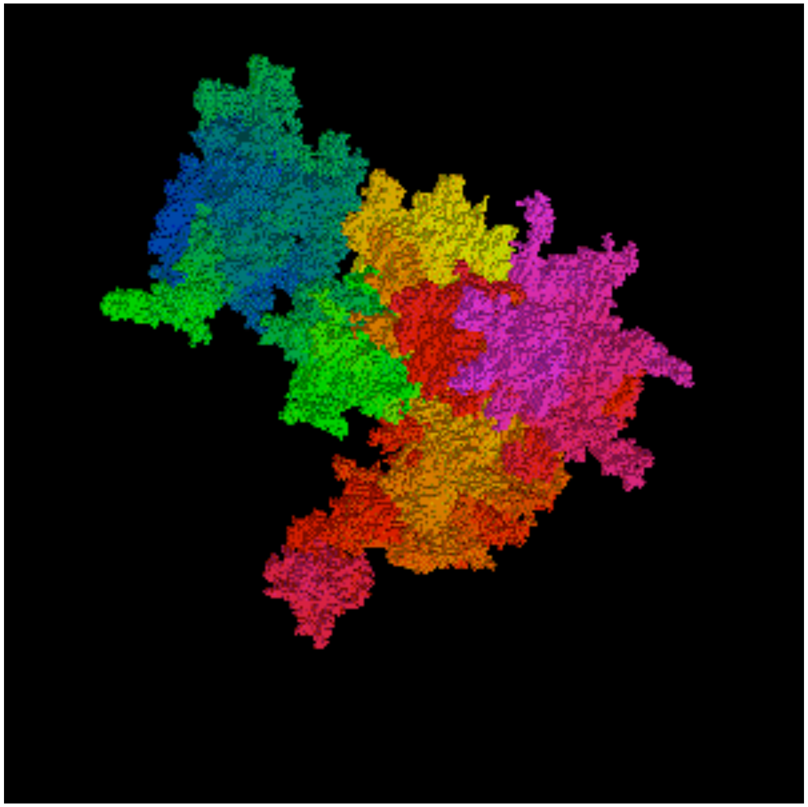
Avalanche Spatial Structure

Beyond Critical Exponents

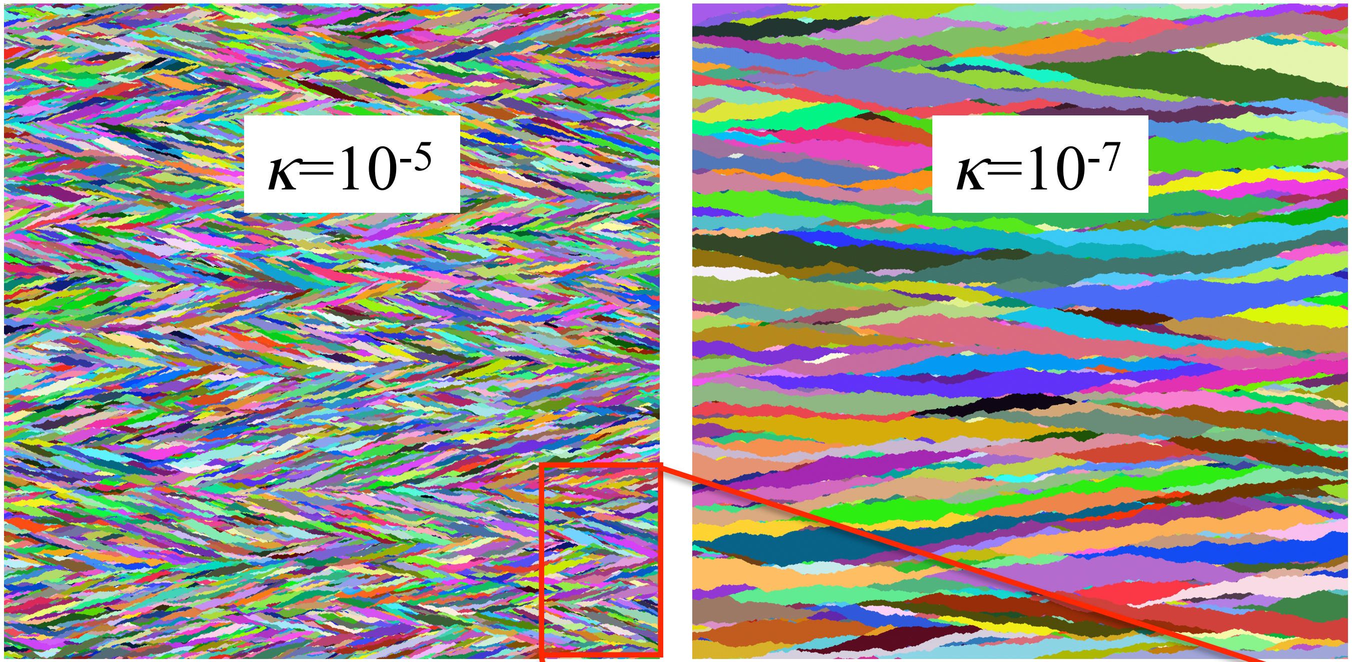


All geometrical features of large avalanches should be universal.

- Correlation functions, fractal dimensions
- Aspect ratios
- Topology (holes, interconnectedness) (string theory)
- Front shapes
- Height, width distributions

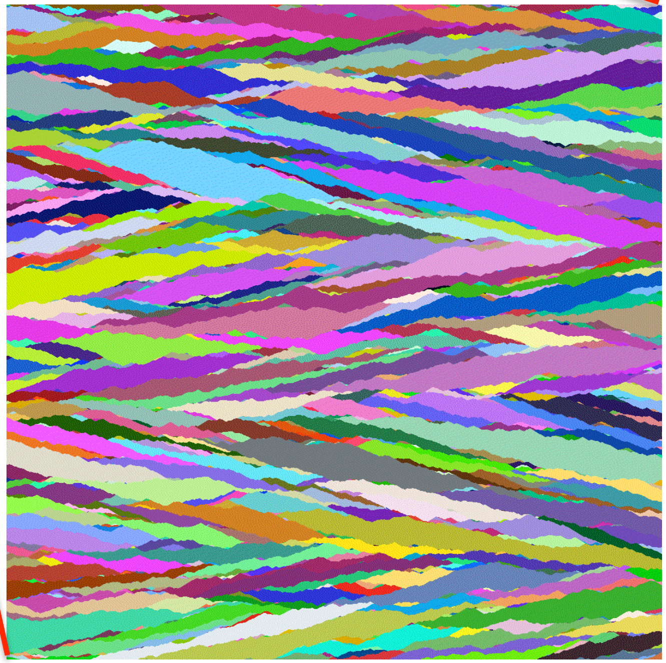


YJ Chen,
Others



Demagnetizing Field κ

Front propagation: limits sizes of avalanches



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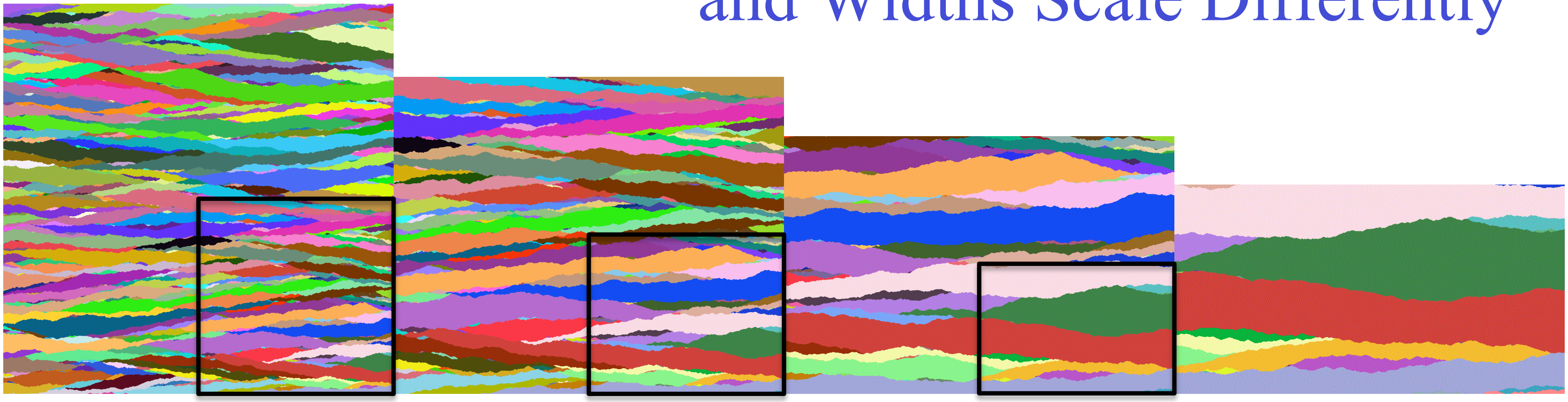
YJ Chen,
Others

Self-Affine

Front propagation: Heights and Widths Scale Differently

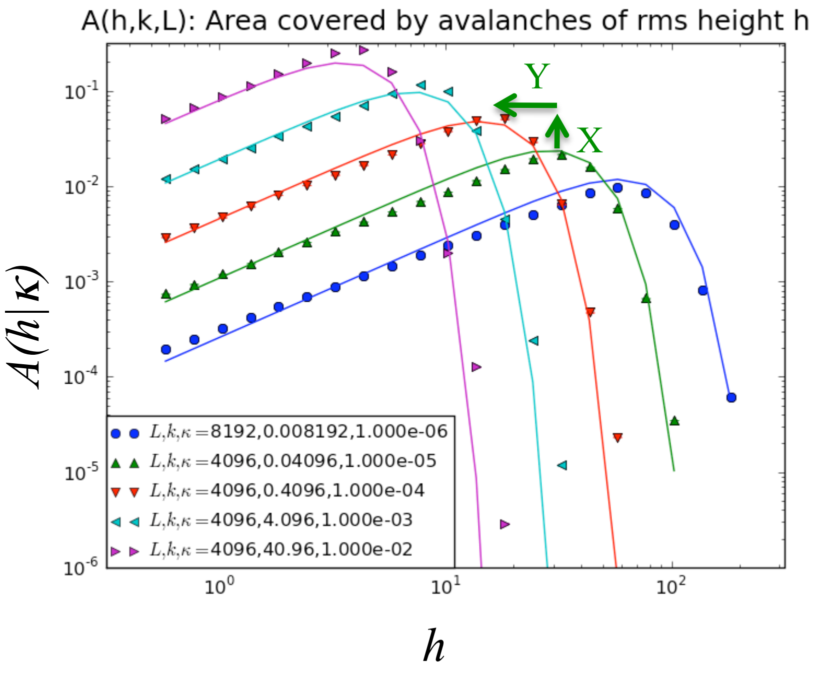


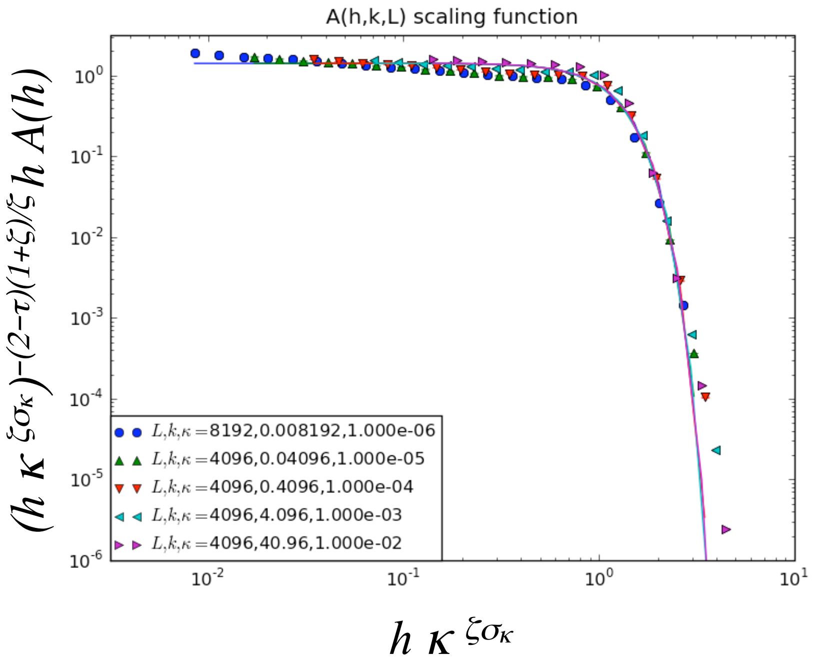
Cut bottom left-hand quarter
Rescale widths by 2, heights by 2^{ζ}
Effective lower demagnetizing field:
    larger avalanches
Fronts appear statistically similar: self-affine

YJ Chen,
Others

Avalanche heights

Scaling away from criticality


A(h,k,L): Area covered by avalanches of rms height h


A(h,k,L) scaling function

$A(h|\kappa)$ similar to $X^{n} A(h / Y_{n} \mid 10^{n}\, \kappa)$

$$A(h|\kappa) = X^{-\log_{10}\kappa}\, \mathbf{A}(h / Y^{-\log_{10}\kappa})$$
$$= \kappa^{-\log_{10} X}\, \mathbf{A}(h\, \kappa^{\log_{10} Y})$$
$$= \kappa^{-(\ldots)}\, \mathbf{A}(h\, \kappa^{\zeta\sigma_\kappa})$$

$\mathbf{A}(h\, \kappa^{\zeta\sigma_\kappa})$ is a *universal scaling function* of the *scaling variable* $h\, \kappa^{\zeta\sigma_\kappa}$.

Traditional Equilibrium Criticality

Energy versus Entropy

Traditional Equilibrium Critical Points

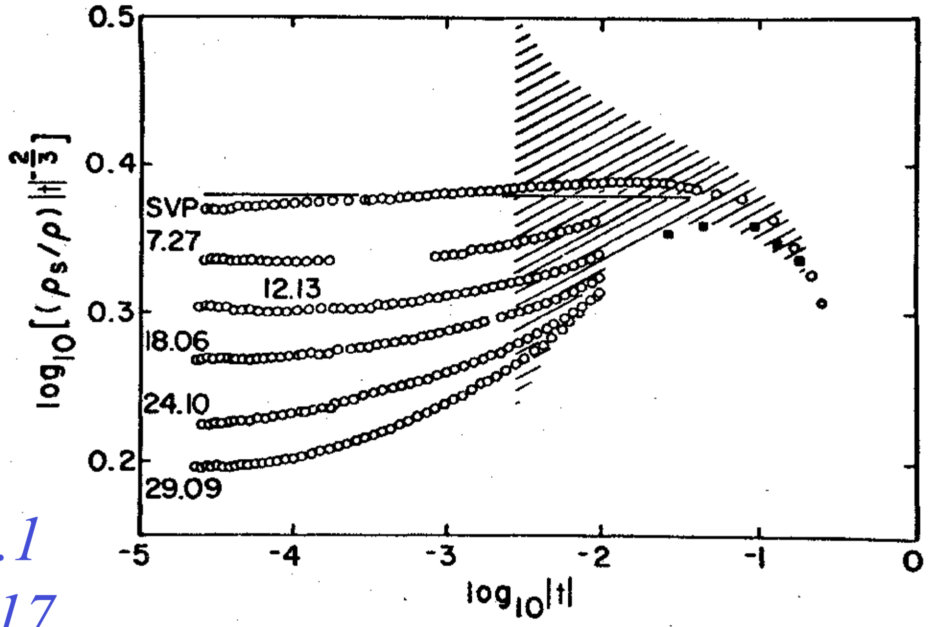
- Ising, Potts (N-state), Heisenberg (3D vector)
- Helium: 3D XY model
- 2D XY Kosterlitz-Thouless Transition, 2D Melting, Hexatic Phases
- Liquid Crystals (Nematic to Smectic A)
- Wetting Transitions

Ahlers: Superfluid Density versus T

- Five decades of $t = |T_c\text{-}T|/T_c$
- Power law ζ
- Singular correction to scaling x

$$\rho_s/\rho = k\,|T_c\text{-}T|^{\zeta}\,(1+d\,|T_c\text{-}T|^{x})$$

- $\zeta_{exp} = 0.6749 \pm 0.0007$, $x_{exp} = 0.5 \pm 0.1$
- $\zeta_{th} = 0.669 \pm 0.002$, $x_{th} = 0.522 \pm 0.017$



Ahlers Rev Mod Phys 52, 489 (1980).
Theory: LeGuillou & Zinn-Justin

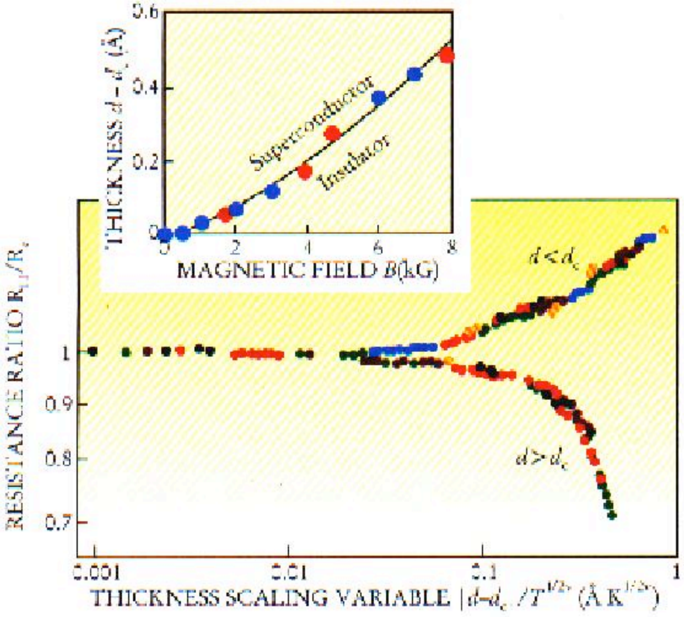
Quantum Phase Transitions

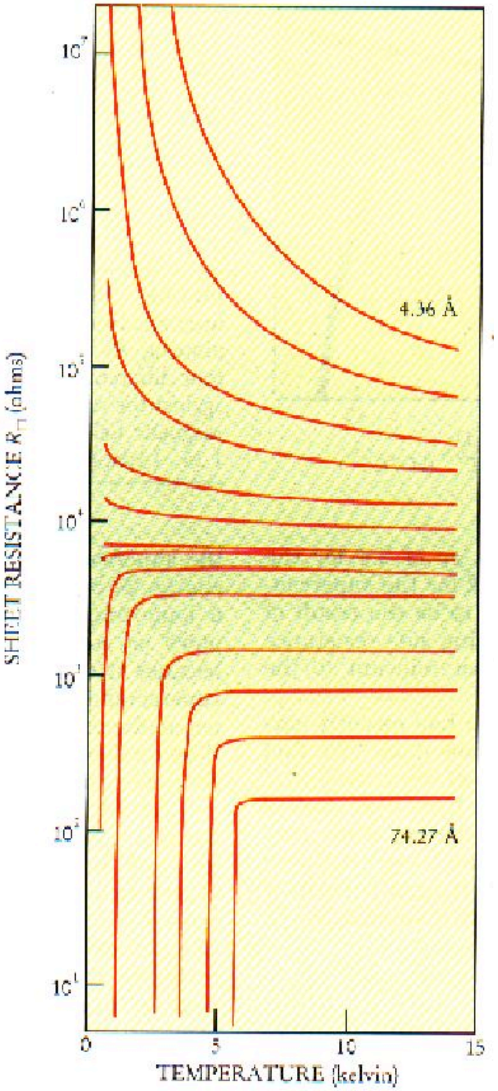
$\hbar$ vs. Disorder, Field, …

Quantum Phase Transitions

- Metal-Insulator Transitions (Localization)
- Superconductor-Insulator Transitions
- Transitions between Quantum Hall Plateaus
- Macroscopic Quantum Tunneling
  (Quantum Coherence and Schrödinger's Cat)
- Kondo Effect

SC to Insulator with
Film *Thickness*
Right: Resistance vs. *T*
Left: Scaling Plot
R/R_c vs. $|d-d_c|/T^{1/z\nu}$
Left Inset: Phase
Boundary *(B, d)*

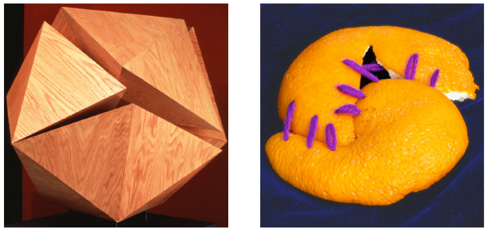




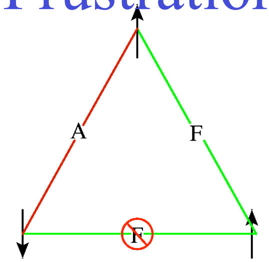
Goldman & Markovic, Phys. Today, p. 39, Nov. 1998

Disordered Systems

Fancy Tools, Still Controversial

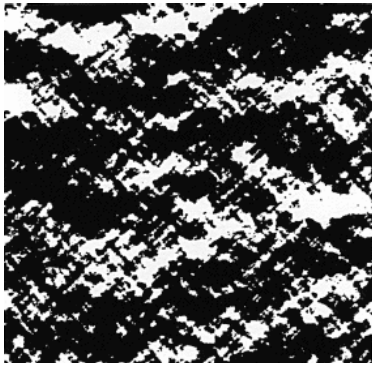


Frustration

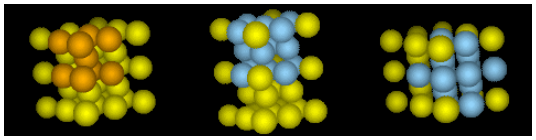


Disordered Systems (Disorder vs. Temperature)

- Spin Glasses: Dilute Magnetic Alloys
  - Frustration, Competing Ferro/Antiferro, RKKY
  - Long-range order in Time $lim(t \Rightarrow \infty) <S_i(t)\ S_i(0)>$
  - Replica Theory vs. Clusters
  - Neural Networks, Tweed in Martensites
- Random Field Ising Models
  - Dimensional Reduction, Supersymmetry ***Wrong***
  - Diverging Barriers, Analogies to Glasses?
- Vortex Glass Transition, …

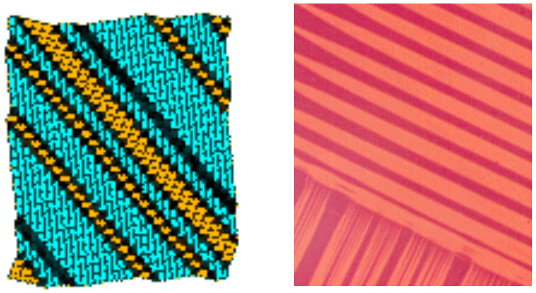


Tweed Precursors



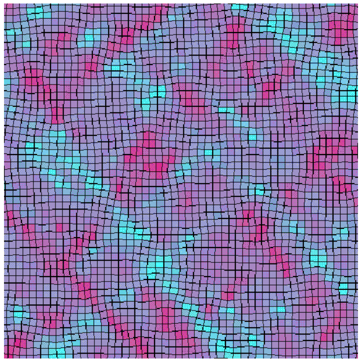
Martensites Change Shape

http://www.lassp.cornell.edu/sethna/Tweed/What_Are_Martensites.html



Form Stripes

But
First
…

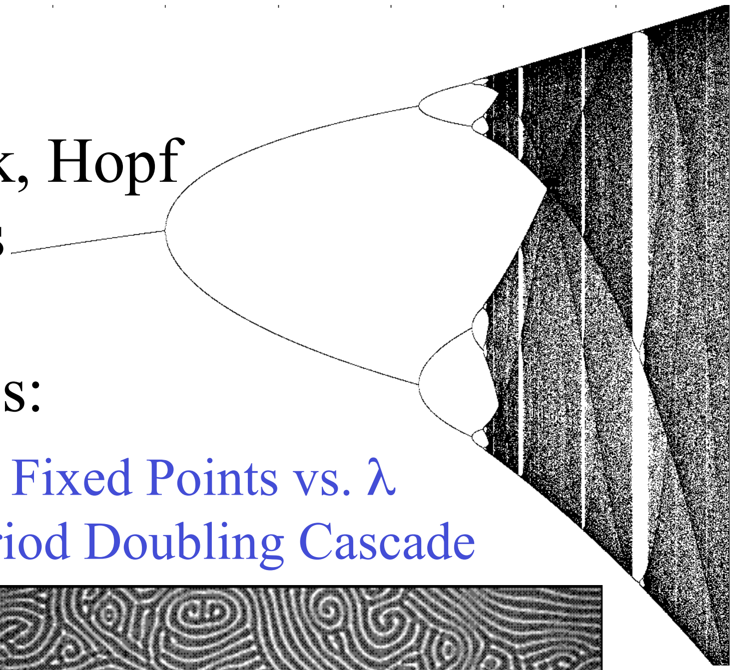


Dynamical Systems and Chaos

Coarse-Graining in Time

Low Dimensional Dynamical Systems

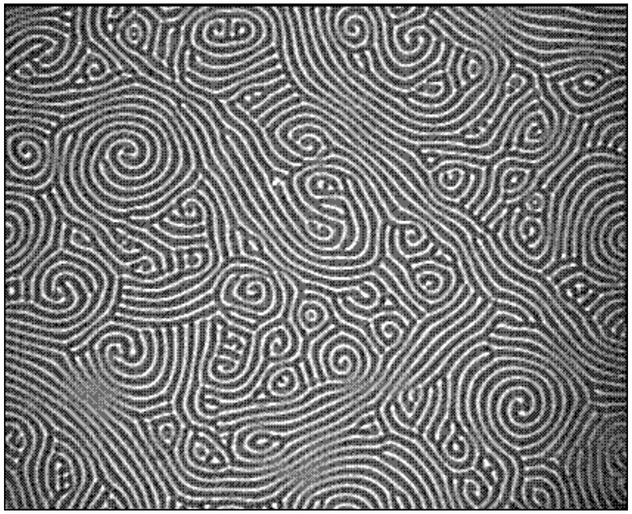
- Bifurcation Theory
  - Saddle-Node, Intermittency, Pitchfork, Hopf
  - Normal Forms = Universality Classes
- Feigenbaum Period Doubling
- Transition from Quasiperiodicity to Chaos: Circle Maps
- Breakdown of the Last KAM Torus: Synchrotrons and the Solar System



Fixed Points vs. λ
Period Doubling Cascade

High-Dimensional Systems

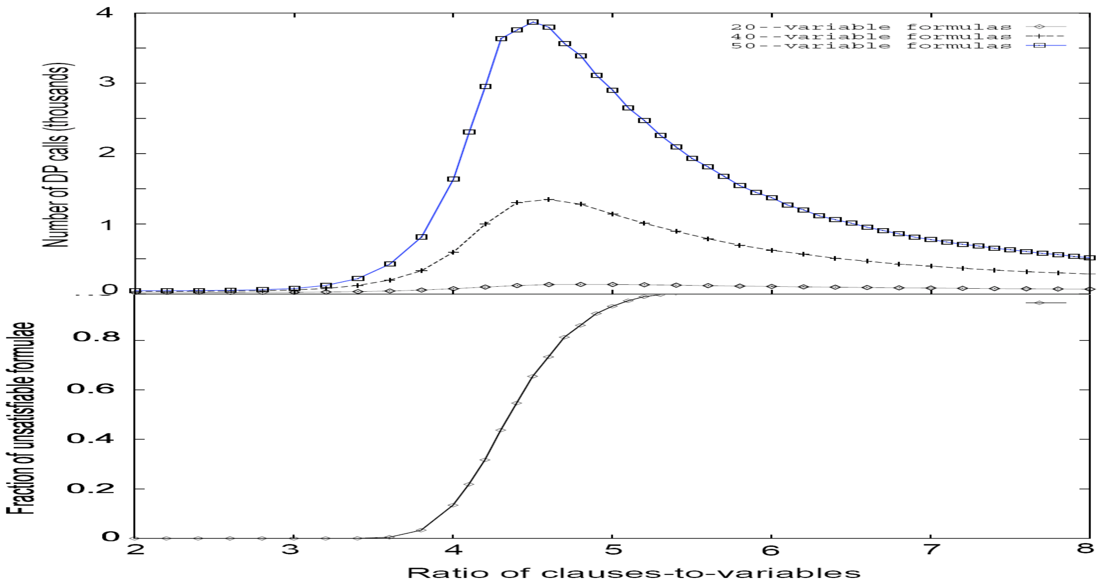
- Turbulence?
- Spatiotemporal Defect Chaos?
- Avalanches…

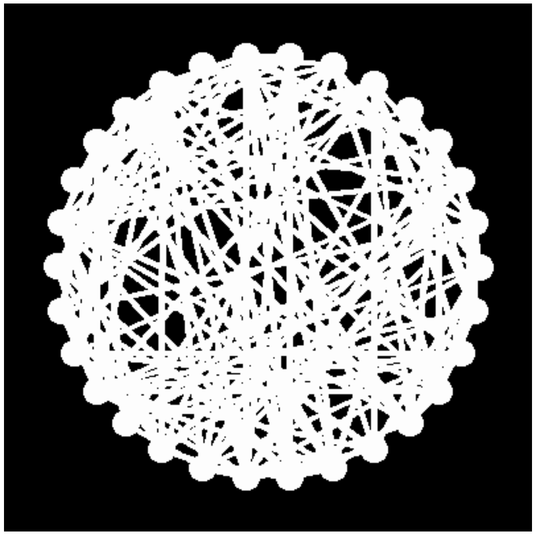


Bodenschatz

Logical Satisfiability and NP-Completeness

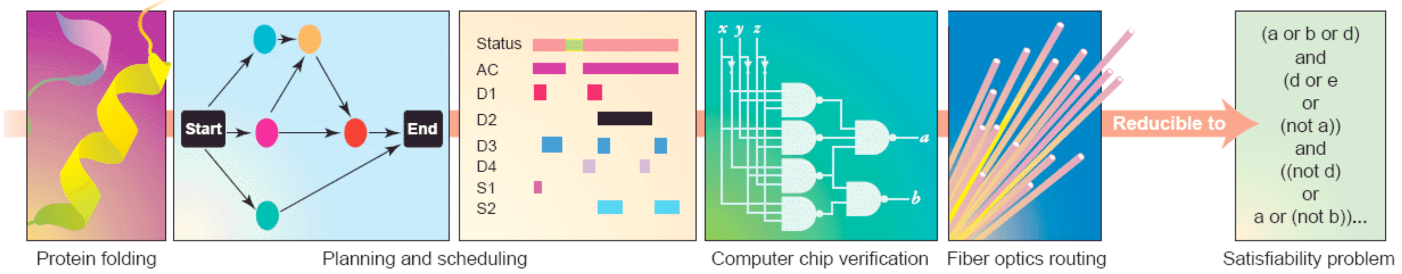
Selman, Kirkpatrick, Gomes, Mézard, Montanari, Monasson, …





Worst-case problems exponentially hard

Typical problem hard only near phase transition



(a or b or d)
and
(d or e
or
(not a))
and
((not d)
or
a or (not b))...

Protein folding

Planning and scheduling

Computer chip verification

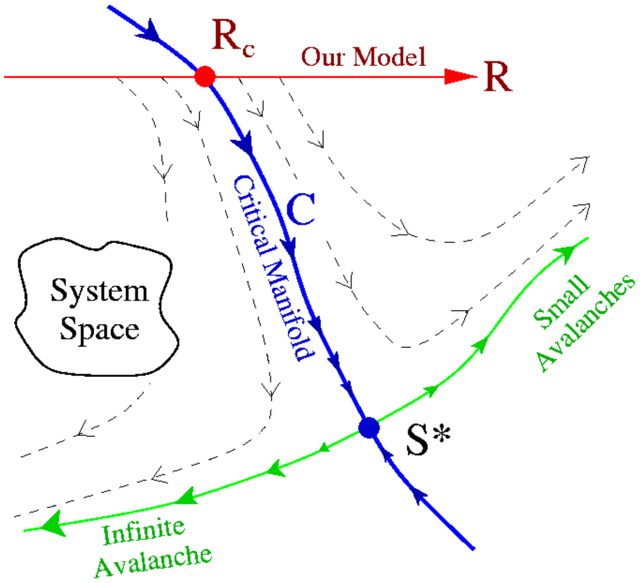
Fiber optics routing

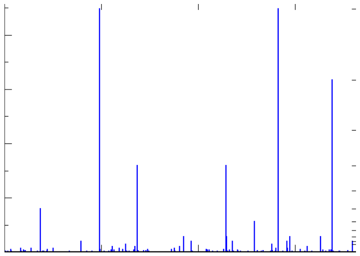
Satisfiability problem

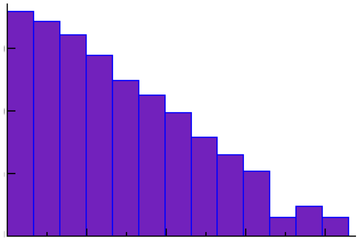
Two phase
transitions!
RG: Coppersmith
Universality?

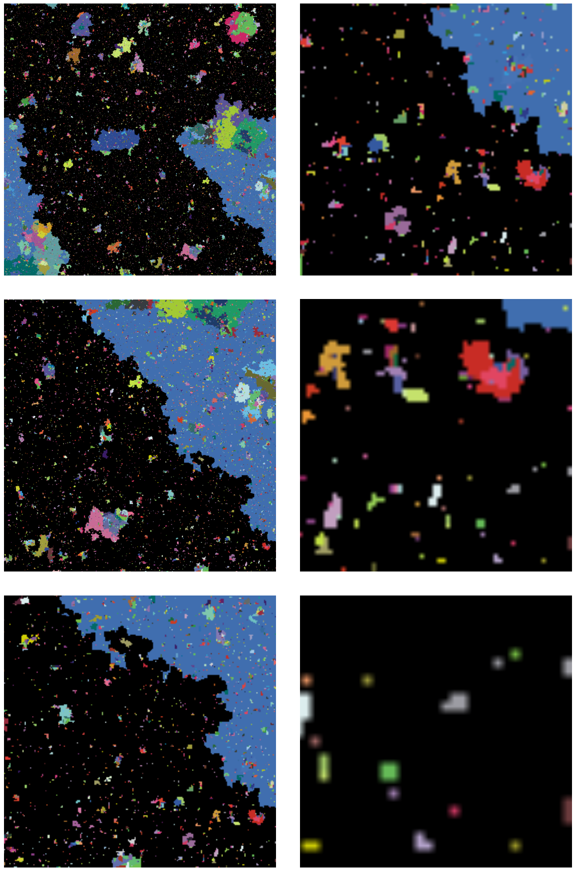
Continuous Phase Transitions

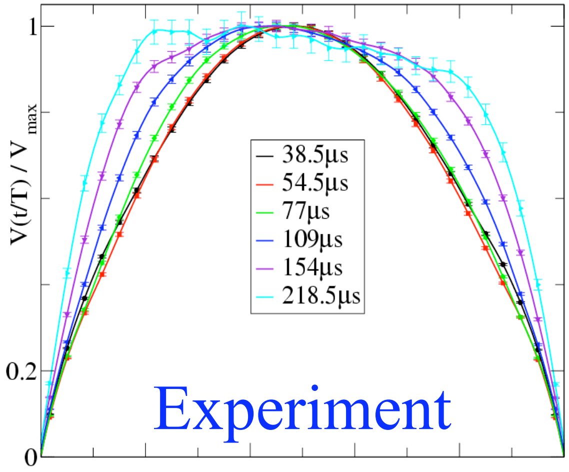
Jim Sethna, Physics 653, Fall 2010

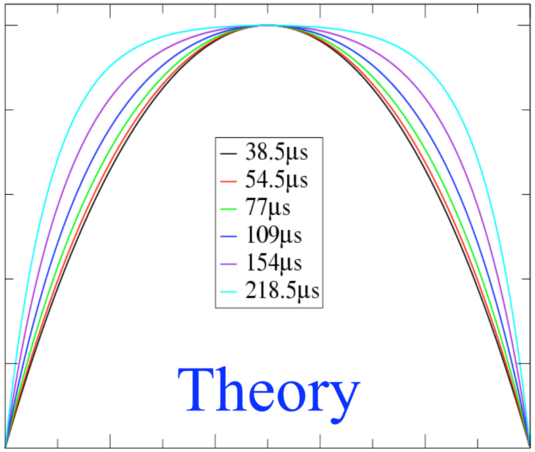












Yanjiun Chen, Stefanos Papanikolaou, Karin Dahmen, Olga Perković, Chris Myers, Matt Kuntz, Gianfranco Durin, Stefano Zapperi, …